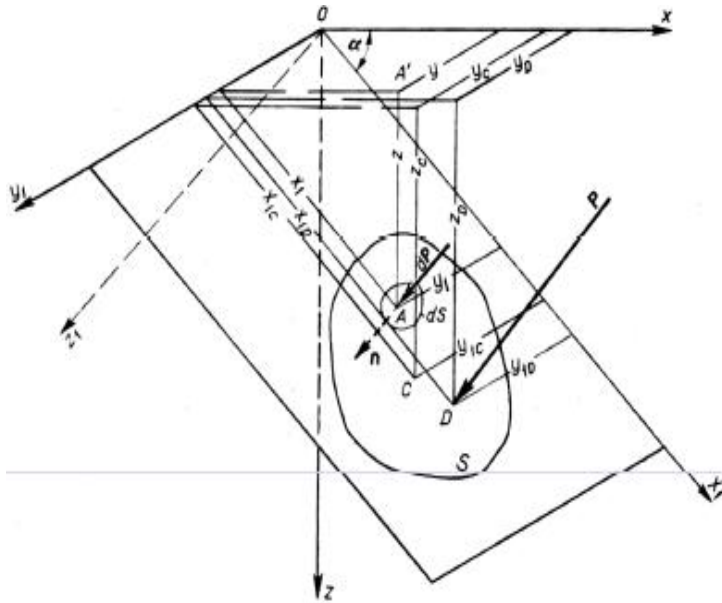


Lecture No. 4

Chapter 4 - Hydrostatic force

Hydrostatic force acting on a flat wall



Hydrostatic force acting on a flat wall has the form of a system of parallel elementary forces perpendicular to the wall. It may be reduced to the resultant force equal to their sum and acting at the centre of action of the parallel forces.

Elementary force:

$$d\bar{P} = \bar{n}(p - p_a)dS = \bar{n}\rho g z dS$$

Total force:

$$\bar{P} = \rho g \int_S \bar{n} z dS = \rho g \bar{n} \int_S z dS = \rho g \bar{n} z_c S$$

where:

z_c

- immersion of the geometric centre of the wall S

The hydrostatic force on the flat wall of an arbitrary outline and arbitrarily inclined to the horizon is equal (in its module) to the weight of the prism of fluid having the base S and height equal to the immersion of the geometrical centre of S under the free surface.

Projections of the hydrostatic force on the axes of the system Oxyz:

$$P_x = \rho g \int_S z \cos(\bar{n}, \bar{i}) dS = -\rho g \sin \alpha \int_S z dS = -\rho g z_C S \sin \alpha$$

$$P_y = \rho g \int_S z \cos(\bar{n}, \bar{j}) dS = 0$$

$$P_z = \rho g \int_S z \cos(\bar{n}, \bar{k}) dS = \rho g \cos \alpha \int_S z dS = \rho g z_C S \cos \alpha$$

Hence the module of the force:

$$P = \sqrt{P_x^2 + P_z^2} = \rho g z_C S$$

Determination of the point of action of the resultant hydrostatic force

Moment of the force:

$$\bar{M}_D = \bar{r}_D \times \bar{P} = \rho g \int_S \bar{r}_D \times \bar{n} z dS = \bar{i}_1 \rho g \int_S y_1 z dS + \bar{j}_1 \rho g \int_S x_1 z dS$$

Projections of the main vector of the moment:

$$x_{1D} P = \rho g \int_S x_1 z dS \qquad y_{1D} P = \rho g \int_S y_1 z dS$$

Co-ordinates of the point of action:

$$x_{1D} = \frac{\int_S x_1 z dS}{\int_S z dS} = \frac{\int_S x_1^2 dS}{\int_S x_1 dS} = \frac{\int_S x_1^2 dS}{x_{1C} S} = \frac{I_{y_1}}{x_{1C} S} \qquad y_{1C} = \frac{\int_S y_1 z dS}{\int_S z dS} = \frac{\int_S x_1 y_1 dS}{\int_S x_1 dS} = \frac{\int_S x_1 y_1 dS}{x_{1C} S} = \frac{D_{z_1}}{x_{1C} S}$$

where it was substituted: $z = x_1 \cos \alpha$

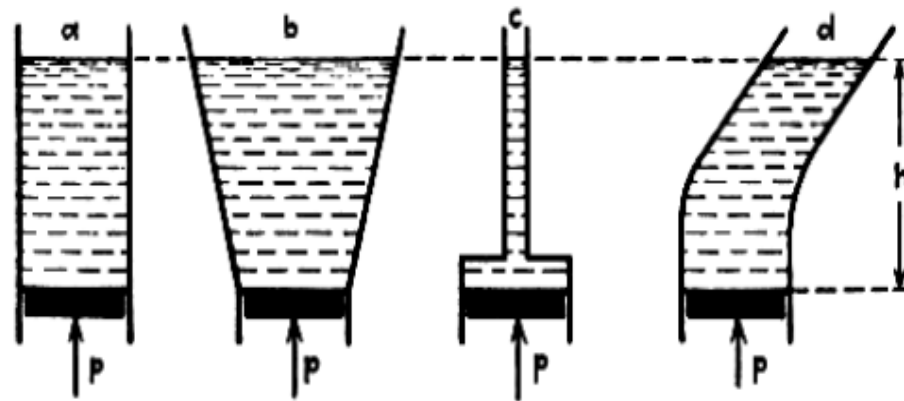
Determination of the point of action of the hydrostatic force requires computation of the moment of inertia of the wall S and computation of its centre of gravity.

Conclusions

Location of the point of action of the hydrostatic force in the system linked to the wall does not depend on the wall inclination.

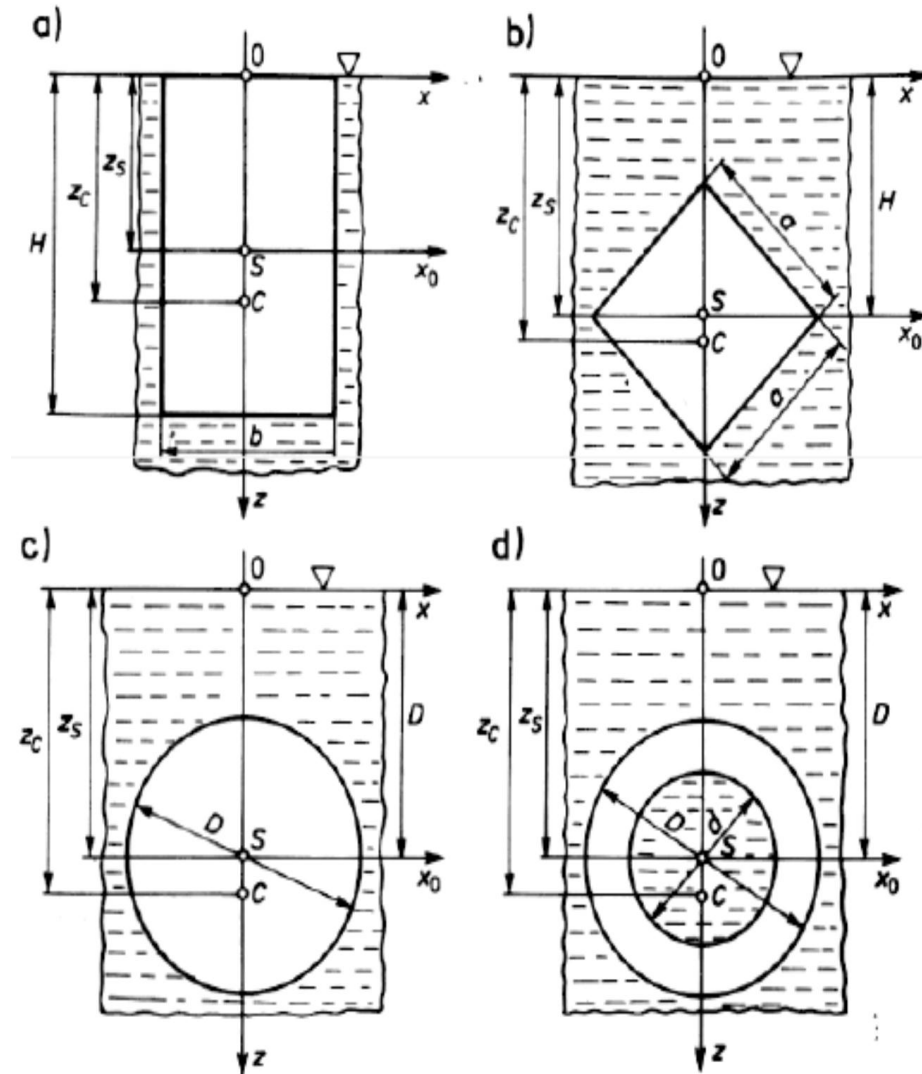
On horizontal and inclined walls the centre of action of the hydrostatic force lies below the geometric centre of the wall.

The magnitude of the hydrostatic force does not depend on the shape of the container.



In all above containers the force on the bottom is the same.

Example No. 1: determine the hydrostatic force and location of its point of action C for vertical walls shown in the picture below.



Solution for the wall „a”

Immersion of the geometric wall centre:

$$z_s = \frac{H}{2}$$

Wall area:

$$S = bH$$

Wall moment of inertia with respect to the main central axis:

$$I_{x0} = \frac{bH^3}{12}$$

Wall moment of inertia with respect to x axis (Steiner's rule):

$$I_x = I_{x0} + z_s^2 S = \frac{bH^3}{12} + \frac{H^2}{4} bH$$

Point of action elevation:

$$z_c = \frac{I_x}{z_s S} = \frac{H}{2} + \frac{H}{6} = \frac{2}{3} H$$

Hydrostatic force module:

$$P = \rho g z_s S = \rho g \frac{H}{2} bH = \frac{\rho g b H^2}{2}$$

Solution for the wall „b”

Immersion of the geometric wall centre:

$$z_S = H$$

Wall area:

$$S = a^2$$

Wall moment of inertia with respect to the main central axis:

$$I_{x0} = \frac{a^4}{12}$$

Wall moment of inertia with respect to x axis (Steiner's rule):

$$I_x = I_{x0} + z_S^2 S = \frac{a^4}{12} + H^2 a^2$$

Point of action elevation:

$$z_c = \frac{I_x}{z_S S} = H + \frac{a^4}{12a^2 H} = H + \frac{a^2}{12H}$$

Hydrostatic force module:

$$P = \rho g z_S S = \rho g H a^2$$

Solution for the wall „c”

Immersion of the geometric wall centre:

$$z_s = D$$

Wall area:

$$S = \frac{\pi D^2}{4}$$

Wall moment of inertia with respect to the main central axis:

$$I_{x0} = \frac{\pi D^4}{64}$$

Wall moment of inertia with respect to x axis (Steiner's rule):

$$I_x = I_{x0} + z_s^2 S = \frac{\pi D^4}{64} + D^2 \frac{\pi D^2}{4}$$

Point of action elevation:

$$z_c = \frac{I_x}{z_s S} = D + \frac{D}{16} = \frac{17}{16} D$$

Hydrostatic force module:

$$P = \rho g z_s S = \rho g D \frac{\pi D^2}{4} = \frac{\pi \rho g D^3}{4}$$

Solution for the wall „d”

Immersion of the geometric wall centre:

$$z_s = D$$

Wall area:

$$S = \frac{\pi(D^2 - d^2)}{4}$$

Wall moment of inertia with respect to the main central axis:

$$I_{x0} = \frac{\pi(D^4 - d^4)}{64}$$

Wall moment of inertia with respect to x axis:

$$I_x = I_{x0} + z_s^2 S = \frac{\pi(D^4 - d^4)}{64} + \frac{\pi(D^2 - d^2)}{4} D^2$$

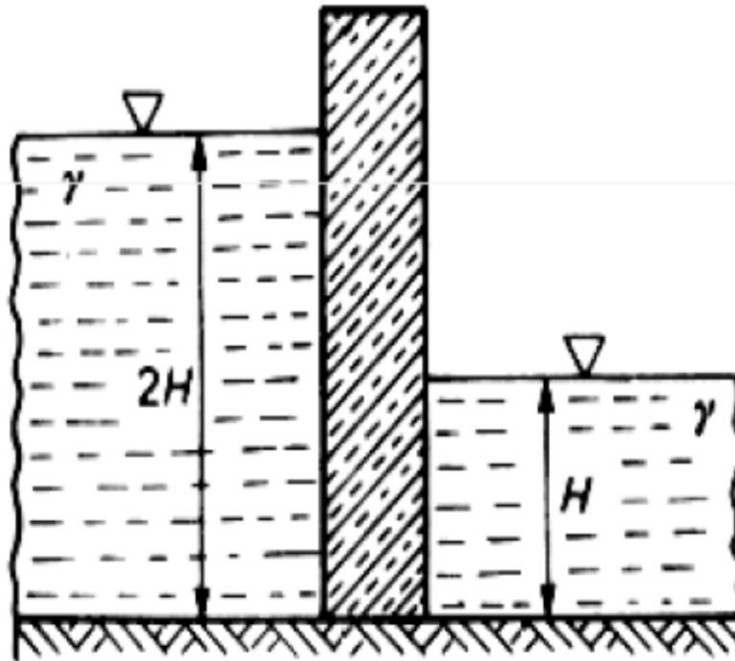
Point of action elevation:

$$z_c = \frac{I_x}{z_s S} = D + \frac{D^2 + d^2}{16D}$$

Hydrostatic force module:

$$P = \rho g z_s S = \rho g \frac{\pi(D^2 - d^2)}{4} D$$

Example No. 2: Determine the hydrostatic moment acting of a vertical dam of width L with respect to its foot. The dam divides a canal of rectangular cross-section. On the left side the liquid level is $2H$, on the right side the liquid level is H .



Solution

Hydrostatic forces on the left and right side are respectively equal to:

$$P_L = \rho g A_L z_{SL}$$

$$P_P = \rho g A_P z_{SP}$$

Assuming the dam width L and knowing that:

$$z_{SL} = H$$

$$z_{SP} = \frac{H}{2}$$

We get:

$$P_L = 2\rho g L H^2$$

$$P_P = \frac{1}{2}\rho g L H^2$$

Points of action of the hydrostatic forces may be determined on the basis of the preceding example for a rectangular wall:

$$z_{CL} = \frac{4}{3}H$$

$$z_{CP} = \frac{2}{3}H$$

Moment acting on the dam is equal to:

$$M = P_L z_L - P_P z_P$$

where:

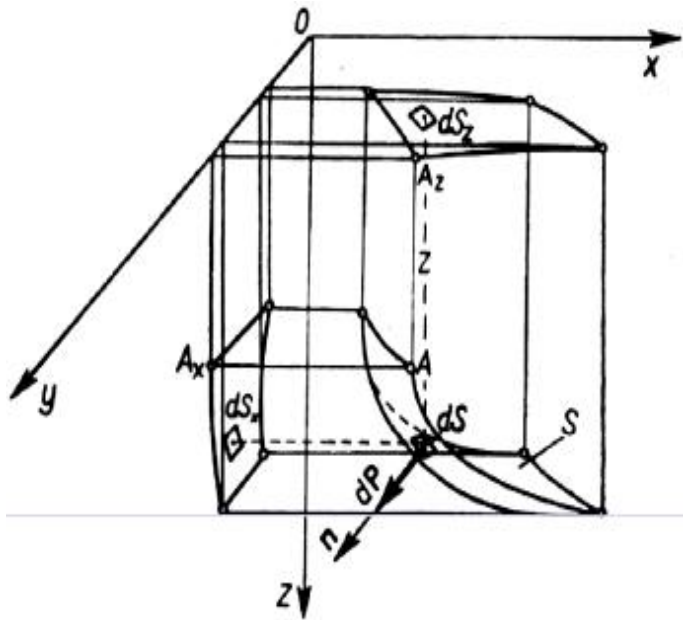
$$z_L = 2H - z_{CL} = 2H - \frac{4}{3}H = \frac{2}{3}H$$

$$z_P = H - z_{CP} = H - \frac{2}{3}H = \frac{1}{3}H$$

Finally, after substitution we get:

$$M = 2\rho g L H^2 \cdot \frac{2}{3}H - \frac{1}{2}\rho g L H^2 \cdot \frac{1}{3}H = \frac{7}{6}\rho g L H^3$$

Hydrostatic force on curved walls



All elementary forces acting on the wall S form a spatial system of forces, which may be reduced to the main force vector and the main moment vector.

Elementary force:

$$d\bar{P} = \bar{n}(p - p_a)dS = \bar{n}\rho g z dS$$

Main force vector:

$$\bar{P} = \rho g \int_S \bar{n} z dS$$

Main moment vector:

$$\bar{M} = \rho g \int_S \bar{r} \times \bar{n} z dS$$

Projections of the main force and moment vectors on the axes Oxyz:

$$P_x = \rho g \int_S z \cos(\bar{n}, \bar{i}) dS = \rho g \int_S z dS_x = \rho g z_{Cx} S_x$$

$$P_y = \rho g \int_S z \cos(\bar{n}, \bar{j}) dS = \rho g \int_S z dS_y = \rho g z_{Cy} S_y$$

$$P_z = \rho g \int_S z \cos(\bar{n}, \bar{k}) dS = \rho g \int_S z dS_z = \rho g V$$

$$M_x = \rho g \int_S z [y \cos(\bar{n}, \bar{k}) - z \cos(\bar{n}, \bar{j})] dS = \rho g \int_S z (y dS_z - z dS_y)$$

$$M_y = \rho g \int_S z [z \cos(\bar{n}, \bar{i}) - x \cos(\bar{n}, \bar{k})] dS = \rho g \int_S z (z dS_x - x dS_z)$$

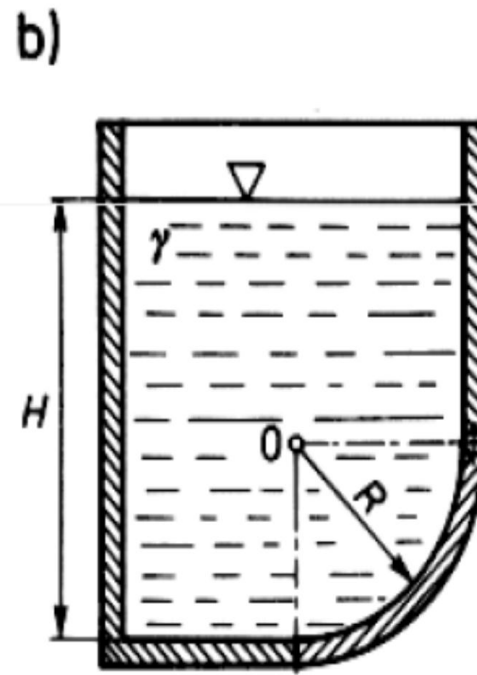
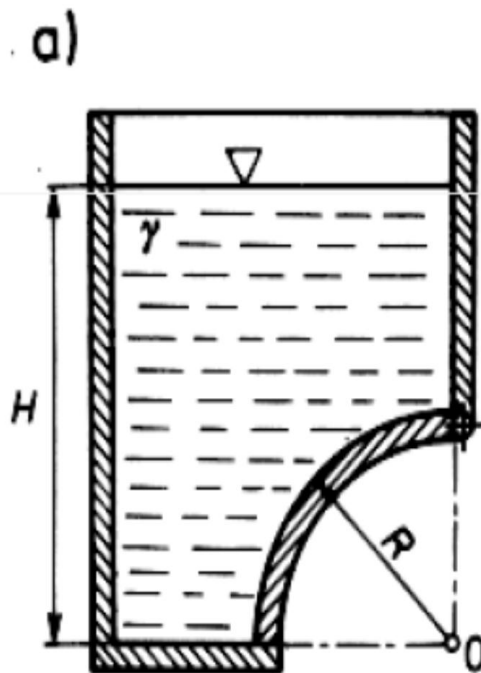
$$M_z = \rho g \int_S z [x \cos(\bar{n}, \bar{j}) - y \cos(\bar{n}, \bar{i})] dS = \rho g \int_S z (z dS_y - y dS_x)$$

Conclusions

Projection of the hydrostatic force on an arbitrary horizontal direction is equal to the total hydrostatic force exerted on a flat wall, the area of which is equal to the projection of the curved wall onto the surface perpendicular to the considered direction. As the areas of the horizontal projections do not depend on the shape of the wall S , but only on its contour, similarly the horizontal projections of the hydrostatic force depend only on the limiting contour of S .

The vertical projection of the hydrostatic force is equal to the weight of the prism of fluid contained between the wall S and its projection onto the free surface.

Example No. 3: a water tank is closed with a rotating flap shaped as a quarter of circular cylinder of radius R and length L . Determine the hydrostatic force acting on the flap in two cases a) and b). Assume water density equal to ρ .



Solution

Horizontal force components are equal in both cases and they are:

$$P_{Xa} = P_{Xb} = \rho g R L \left(H - \frac{R}{2} \right)$$

Vertical force components are respectively equal to:

$$P_{Za} = \rho g H R L - \rho g L \frac{\pi R^2}{4} = \rho g L R \left(H - \frac{\pi R}{4} \right)$$

$$P_{Zb} = \rho g H R L - \left(\rho g L R^2 - \rho g L \frac{\pi R^2}{4} \right) = \rho g R L \left(H - R + \frac{\pi R}{4} \right)$$

The resultant forces are respectively equal to:

$$P_a = \sqrt{P_{Xa}^2 + P_{Za}^2}$$

$$P_b = \sqrt{P_{Xb}^2 + P_{Zb}^2}$$

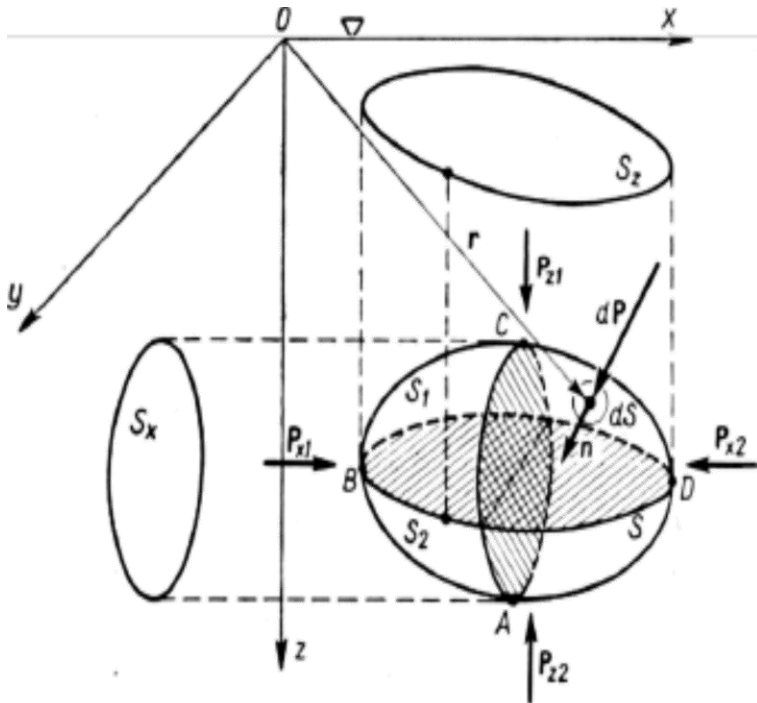
They are inclined to the horizon by an angle: $\alpha = \arctg \frac{P_x}{P_z}$

Chapter 5 – Floating of bodies. The Archimedes theorem

An elementary hydrostatic force $d\bar{P} = \bar{n}\rho g z dS$ acts on every element dS

The total hydrostatic force:
$$\bar{P} = \rho g \int_S \bar{n} z dS$$

The main moment vector of this force:
$$\bar{M} = \rho g \int_S \bar{r} \times \bar{n} z dS$$



The horizontal components of the hydrostatic force are equal zero. The total hydrostatic force is reduced to the vertical force acting on two parts of the surface joined at the common contour: the lower part BAD and the upper part BCD.

Force at the lower surface

$$P_{z1} = \rho g \int_{S_1} z dS = \rho g V_1$$

Force at the upper surface

$$P_{z2} = \rho g \int_{S_2} z dS = \rho g V_2$$

Resultant force

$$P_z = P_{z1} - P_{z2} = -\rho g (V_2 - V_1) = -\rho g V$$

Finally the buoyancy force

$$W = -P_z = \rho g V$$

The hydrostatic buoyancy force acting on the body submerged in a liquid is equal to the weight of the liquid displaced by this body. The line of action of the buoyancy force passes through the centre of mass of the displaced liquid, or the centre of buoyancy.

Determination of the line of action of the buoyancy force.

Components of the main moment of the buoyancy force:

$$M_x = \rho g \int_S z(ydS_z - zdS_y) = \rho g \int_V \frac{\partial(yz)}{\partial z} dV - \rho g \int_V \frac{\partial z^2}{\partial y} dV = \rho g \int_V ydV = \rho g y_C V = y_C P_z$$

where: $y_C = \frac{1}{V} \int_V ydV$ is the y coordinate of the centre of volume V

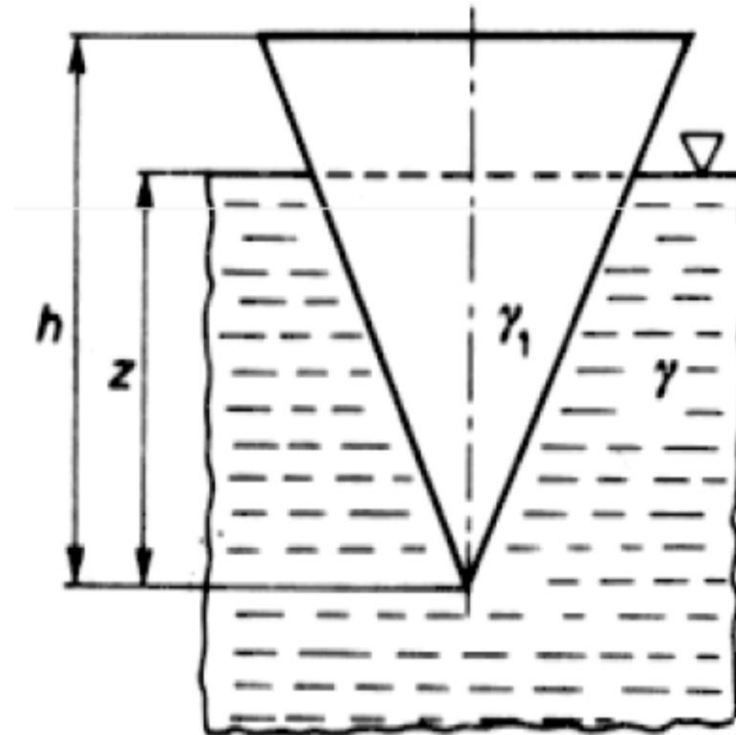
$$M_y = \rho g \int_S z(zdS_x - xdS_z) = \rho g \int_V \frac{\partial z^2}{\partial z} dV - \rho g \int_V \frac{\partial(xz)}{\partial z} dV = -\rho g \int_V xdV =$$
$$= -\rho g x_C V = -x_C P_z$$

where: $x_C = \frac{1}{V} \int_V xdV$ is the x co-ordinate of the centre of volume V

The line of action of the hydrostatic buoyancy force is directed vertically upwards and passes through the point $x_C y_C$

Example No. 1:

A cube of height h made of material of specific weight γ_1 floats in a liquid with its vertex down. Determine the cube immersion if the specific weight of the liquid is equal to γ .



If the base area of the cube is A_h , and the waterline area is A_z

then the gravity force is equal to:

$$G = \frac{1}{3} A_h h \gamma_1$$

And the buoyancy force is:

$$W = \frac{1}{3} A_z z \gamma$$

It follows from the equilibrium condition:

$$G = W \rightarrow \gamma_1 A_h h = \gamma A_z z \rightarrow z = h \frac{A_h}{A_z} \frac{\gamma_1}{\gamma}$$

As there is:

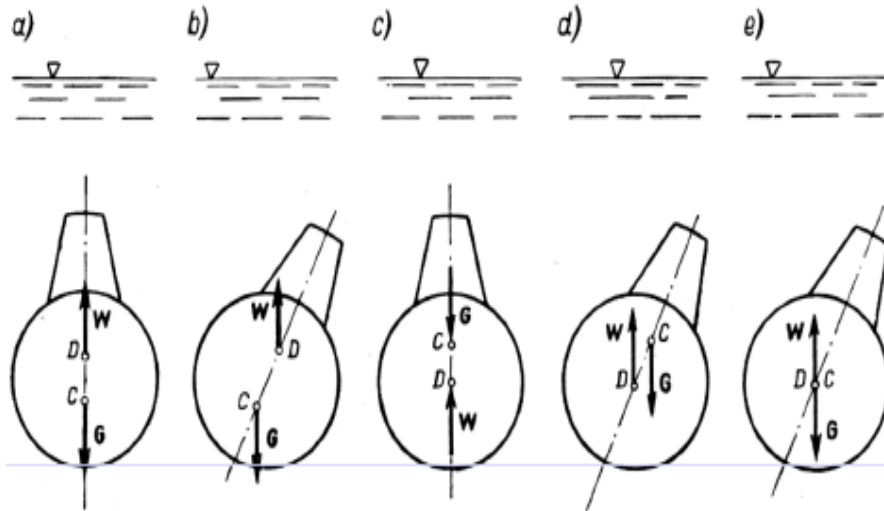
$$\frac{A_h}{A_z} = \frac{h^2}{z^2}$$

finally we have:

$$z = h \sqrt[3]{\frac{\gamma_1}{\gamma}}$$

Stability of the floating bodies

Stability of the completely submerged body



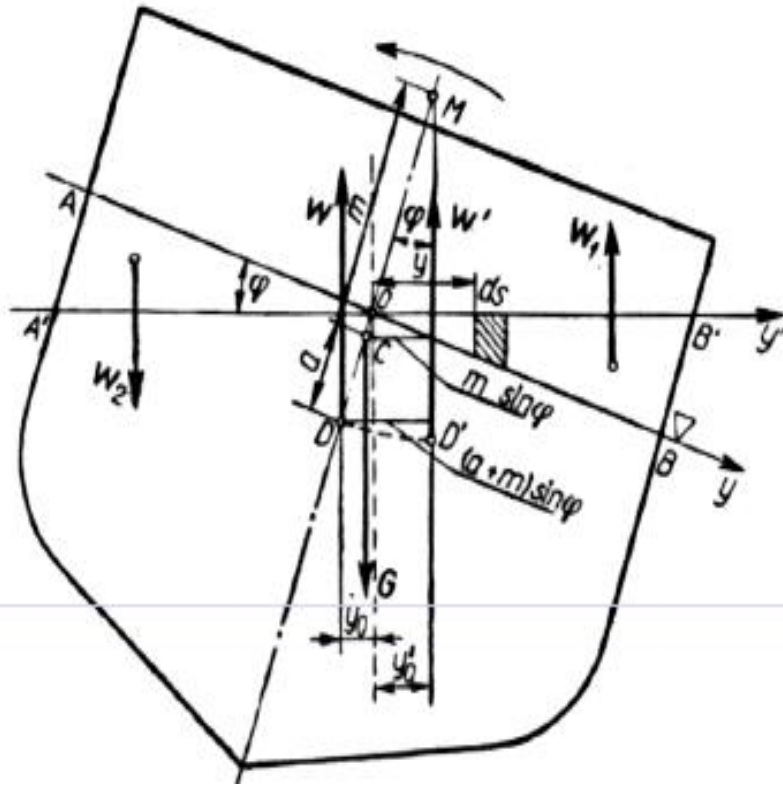
Stable equilibrium – Centre of buoyancy is located above the centre of gravity – Figs a) and b). Deflection from the equilibrium position generates the moment of the pair of forces acting against the deflection and enforcing return to the previous position.

Unstable equilibrium – centre of buoyancy is located below the centre of gravity – Figs c) and d). Deflection from the equilibrium position generates the moment of the pair of forces acting to increase the deflection.

Neutral equilibrium – Fig e) – in an arbitrary position of the body the forces of buoyancy and gravity balance each other without producing the moment which influences position of the body.

Conclusion: in the case of completely submerged body the stable equilibrium is ensured when the centre of buoyancy is located above the centre of gravity.

Stability of the partly submerged body



Assumption: the inclination angle is small

In the case of a partly submerged body the centre of buoyancy changes its location when inclined. The analysis of stability is based on determination of the position of the centre of buoyancy after the body is inclined by an angle φ .

The heeling moment:

$$M_0 = G a \sin \varphi$$

$$G = \rho g V$$

-weight of the body

$$M_0 = \rho g a V \sin \varphi$$

The righting moment

$$dM_1 = ydW_1 \quad \text{the right hand side of the inclined body}$$

$$dW_1 = \rho g y dS \sin \varphi$$

$$dM_1 = \rho g y^2 dS \sin \varphi$$

$$M_1 = \rho g \sin \varphi \int_{S_1} y^2 dS \quad M_2 = \rho g \sin \varphi \int_{S_2} y^2 dS$$

$$M = M_1 + M_2 = \rho g \sin \varphi \int_S y^2 dS = \rho g I_x \sin \varphi$$

The metacentric height m is defined (see the sketch):

$$M - M_0 = \rho g I_x \sin \varphi - \rho g a V \sin \varphi$$

$$m = \frac{I_x}{V} - a$$

I_x - moment of inertia of the waterline

$S = S_1 + S_2$ - area of the waterline

V - volume of the submerged part of the body

There are three possible cases:

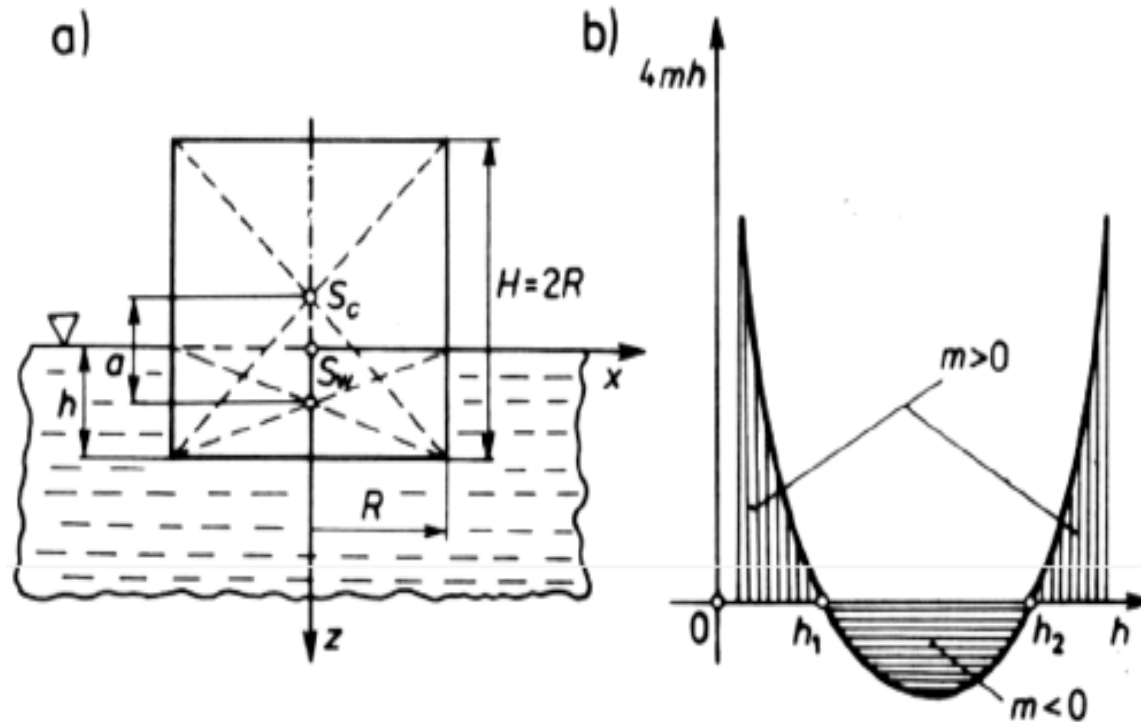
The metacentric height is positive – **stable equilibrium** - by deflection from the equilibrium position the righting moment of pair of forces is generated.

The metacentric height is equal zero – **neutral equilibrium**

The metacentric height is negative – **unstable equilibrium** – by deflection from the position of equilibrium the moment of the pair of forces acts to increase the deflection.

Conclusion: the partly submerged body may be in the state of stable equilibrium even if its centre of gravity is located above the centre of buoyancy. If the moment of inertia of the waterline is larger (the body is „wider”), the centre of gravity may be located higher.

Example No. 2



A circular cylinder having the base radius R and height $H=2R$ floats in an upright position (Fig. a). The centre of gravity of the cylinder coincides with its geometrical centre. For which submergence of the cylinder h its equilibrium is stable?

The waterline of the cylinder is a circle, the inertia moment of which is:

$$I_x = \frac{\pi R^4}{4}$$

The volume of liquid displaced by the cylinder is:

$$V = \pi R^2 h$$

The distance between the centre of gravity of the cylinder and the centre of buoyancy is:

$$a = \frac{H}{2} - \frac{h}{2} = R - \frac{h}{2}$$

Substitution of the above relations to the formula for the metacentric height leads to:

$$m = \frac{I_x}{V} - a = \frac{R^2}{4h} - R + \frac{h}{2}$$

This may be transformed into:

$$4mh = R^2 - 4Rh + 2h^2$$

This function may be plotted as a parabola (Fig. b), the zero points of which may be determined by equating the right hand side to zero:

$$2h^2 - 4Rh + R^2 = 0$$

The above quadratic equation has two solutions:

$$h_1 = \frac{2 - \sqrt{2}}{2} R \approx 0,29R$$

$$h_2 = \frac{2 + \sqrt{2}}{2} R \approx 1,7R$$

The diagram in Fig. B shows, that the stable equilibrium of the cylinder is achieved at the shallow draught according to the condition:

$$h < 0,29R$$

and at the deep draught:

$$h > 1,7R$$

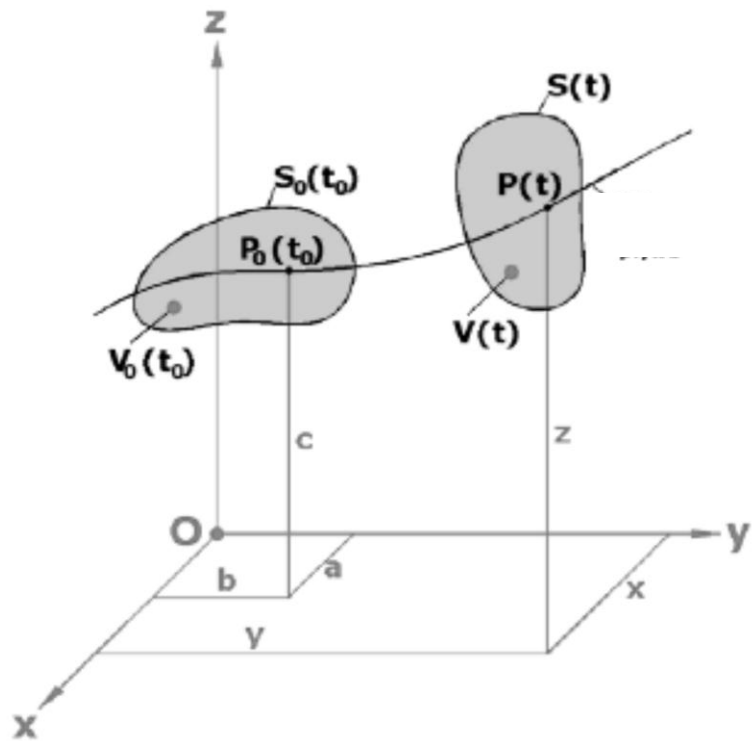
For the intermediate draughts of the cylinder its equilibrium is unstable.

Chapter 6: Kinematics of fluids – Lagrange and Euler approach – Stream lines – Paths of fluid elements

Kinematyka płynów – Podejście Lagrange’a i Euler’a – Linie prądu – Tory elementów płynu

Lagrange approach is based on description of the spatial motion of a certain separated mass of fluid composed always of the same molecules.

Podejście Lagrange’a (inaczej metoda wędrówna) polega na opisywaniu ruchu w przestrzeni pewnej wydzielonej masy płynu składającej się zawsze z tych samych molekuł.



V – volume of a certain mass of fluid (fluid volume) surrounded by the surface S , which is impenetrable for the fluid elements.

The mass of fluid moves from the location V_0 at time t_0 to the location V in time t .

V - objętość pewnej masy płynu (objętość płynna) otoczona powierzchnią S , która jest nieprzenikliwa dla elementów płynu.

Masa płynu przemieszcza się od położenia V_0 w chwili t_0 do położenia V w chwili t .

The fluid element P , constituting a part of the volume V , moves in space tracing the path, which may be described by the following set of parametric equations with time t as the parameter:

Element płynu P stanowiący część objętości V przemieszcza zakreślając w przestrzeni tor elementu, który może być opisany równaniami parametrycznymi z czasem t jako parametrem:

$$x = x(a, b, c, t)$$

$$y = y(a, b, c, t)$$

$$z = z(a, b, c, t)$$

Changing the quantities a , b and c in the equations we describe different fluid elements.

Zmieniając w równaniach wielkości a , b i c opisujemy coraz to inne elementy płynu

The parameters describing fluid motion depend in the same way on a , b , c , t .

Wielkości opisujące ruch płynu są w taki sam sposób zależne od a , b , c , t .

$$\bar{u} = \bar{u}(a, b, c, t)$$

$$p = p(a, b, c, t)$$

$$\rho = \rho(a, b, c, t)$$

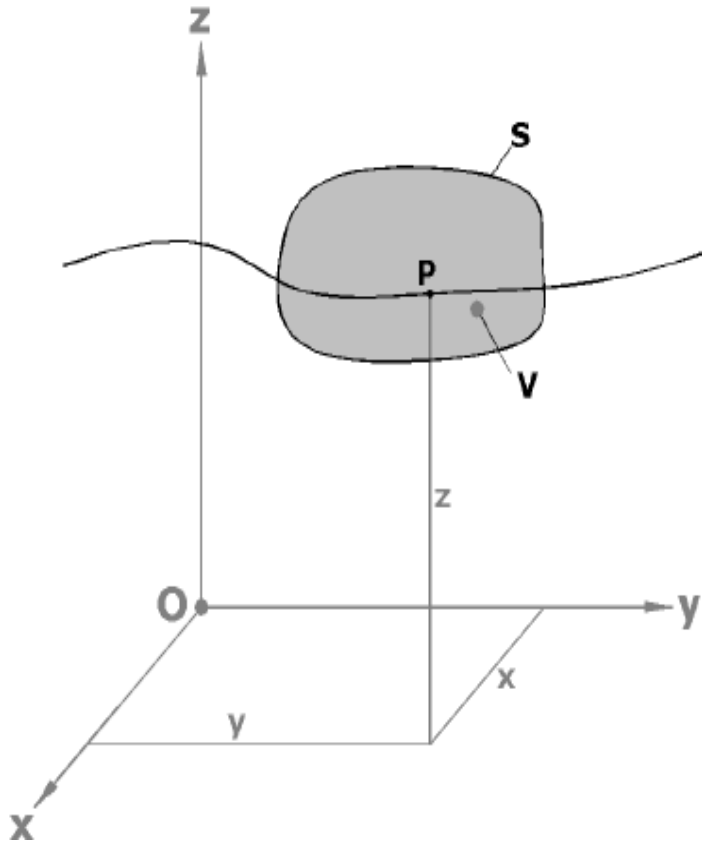
where:
gdzie:

$$\bar{u} = \bar{i}u_x + \bar{j}u_y + \bar{k}u_z$$

$$u_x = \frac{dx}{dt} \quad u_y = \frac{dy}{dt} \quad u_z = \frac{dz}{dt}$$

Euler method is based on selection of the stationary control volume V surrounded by the control surface S . Different fluid elements flow through this volume (crossing the surface S) with different values of velocity, pressure, density etc. The values of these quantities in different points of the control volume are the subject of the description.

Metoda Euler'a (metoda lokalna) polega na wybraniu w przestrzeni nieruchomej objętości kontrolnej V ograniczonej powierzchnią kontrolną S . Przez tę objętość przepływają kolejno różne elementy płynu z różnymi wartościami takich wielkości jak prędkość, ciśnienie, gęstość itd. Przedmiotem opisu są wartości tych wielkości w wybranych punktach objętości kontrolnej.



$$\bar{u} = \bar{u}(x, y, z, t)$$

$$p = p(x, y, z, t)$$

$$\rho = \rho(x, y, z, t)$$

where:

gdzie:

$$\bar{u} = \bar{i}u_x(x, y, z, t) + \bar{j}u_y(x, y, z, t) + \bar{k}u_z(x, y, z, t)$$

The material derivative

The material derivative is a particular interpretation of the complete derivative of a function of several variables, related to the Eulerian description of the fluid motion. It shows, how an arbitrary flow parameter describing the fluid element changes with time when the element is moving in the field of this parameter. It is explained below using the example of an arbitrary scalar parameter H , which is a direct and involved function of time. If H is the function of Euler variables, then we have:

Pochodna materialna (substancjalna)

Pochodna materialna jest szczególną interpretacją pochodnej funkcji wielu zmiennych, związaną z eulerowskim sposobem opisu ruchu płynu. Pokazuje ona, w jaki sposób zmienia się w czasie dowolny parametr charakteryzujący element płynu poruszający się w polu tego parametru. Wyjaśnimy to na przykładzie dowolnego parametru skalarnego H , będącego jawną i złożoną funkcją czasu. Jeżeli H jest funkcją zmiennych Eulera to mamy:

$$H = H(t, x(t), y(t), z(t))$$

Following the definition of the complete differential we have:

Zgodnie z definicją różniczki zupełnej funkcji wielu zmiennych mamy:

$$\frac{DH}{Dt} = \frac{\partial H}{\partial t} + \frac{\partial H}{\partial x} \frac{dx}{dt} + \frac{\partial H}{\partial y} \frac{dy}{dt} + \frac{\partial H}{\partial z} \frac{dz}{dt}$$

but:
Ale mamy:

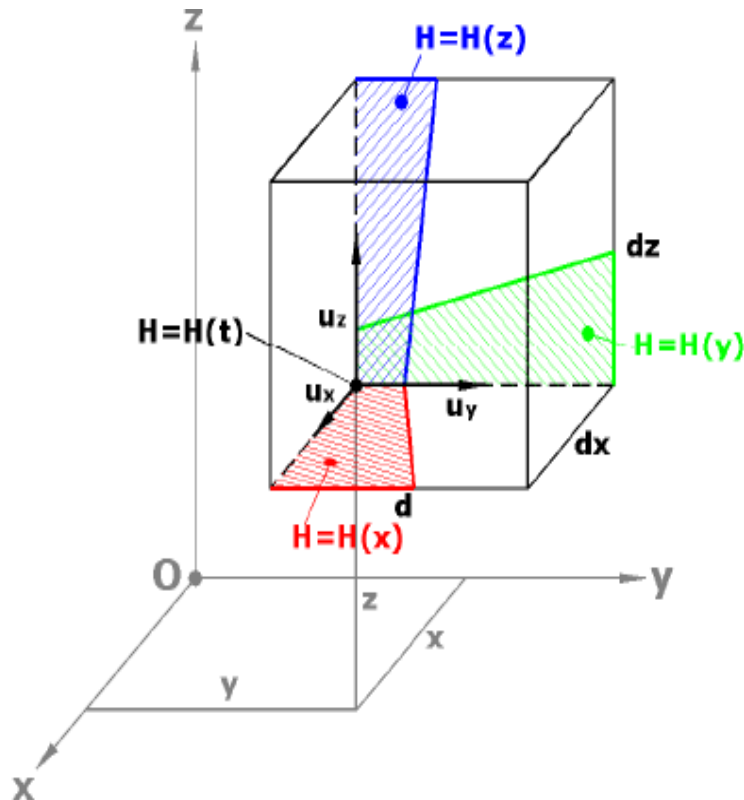
$$\frac{dx}{dt} = u_x \quad \frac{dy}{dt} = u_y \quad \frac{dz}{dt} = u_z$$

what leads to:
co prowadzi do:

$$\frac{DH}{Dt} = \frac{\partial H}{\partial t} + \frac{\partial H}{\partial x} u_x + \frac{\partial H}{\partial y} u_y + \frac{\partial H}{\partial z} u_z = \frac{\partial H}{\partial t} + \bar{u} \bullet \nabla H = \frac{\partial H}{\partial t} + \bar{u} \bullet \text{grad} H$$

Material derivative = local derivative + convection derivative

Pochodna materialna = pochodna lokalna + pochodna unoszenia



The local derivative shows the change of the parameter H with time in the point (x, y, z) , resulting from the unsteadiness of the field H .

The convective derivative shows the change of the parameter H with time, resulting from the motion of the fluid element with velocity u from the point of one value of H to the point of another value of H .

Pochodna lokalna pokazuje zmianę parametru H w czasie w punkcie (x, y, z) wynikającą z niestacjonarności pola H . Pochodna unoszenia pokazuje zmianę parametru H w czasie na skutek przemieszczenia się elementu płynu z prędkością u z punktu o jednej wartości H do punktu o innej wartości H .

Application of the material derivative operator to the components of the velocity field enables calculation of the material acceleration, i.e. the acceleration of the fluid element moving in the unsteady and non-uniform field of flow.

Zastosowanie operatora pochodnej materialnej do składowych pola prędkości pozwala obliczyć przyspieszenie materialne, czyli przyspieszenie elementu płynu poruszającego się w niestacjonarnym i niejednorodnym polu prędkości.

$$\frac{Du_x}{Dt} = \frac{\partial u_x}{\partial t} + u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} + u_z \frac{\partial u_x}{\partial z} = a_x$$

$$\frac{Du_y}{Dt} = \frac{\partial u_y}{\partial t} + u_x \frac{\partial u_y}{\partial x} + u_y \frac{\partial u_y}{\partial y} + u_z \frac{\partial u_y}{\partial z} = a_y$$

$$\frac{Du_z}{Dt} = \frac{\partial u_z}{\partial t} + u_x \frac{\partial u_z}{\partial x} + u_y \frac{\partial u_z}{\partial y} + u_z \frac{\partial u_z}{\partial z} = a_z$$

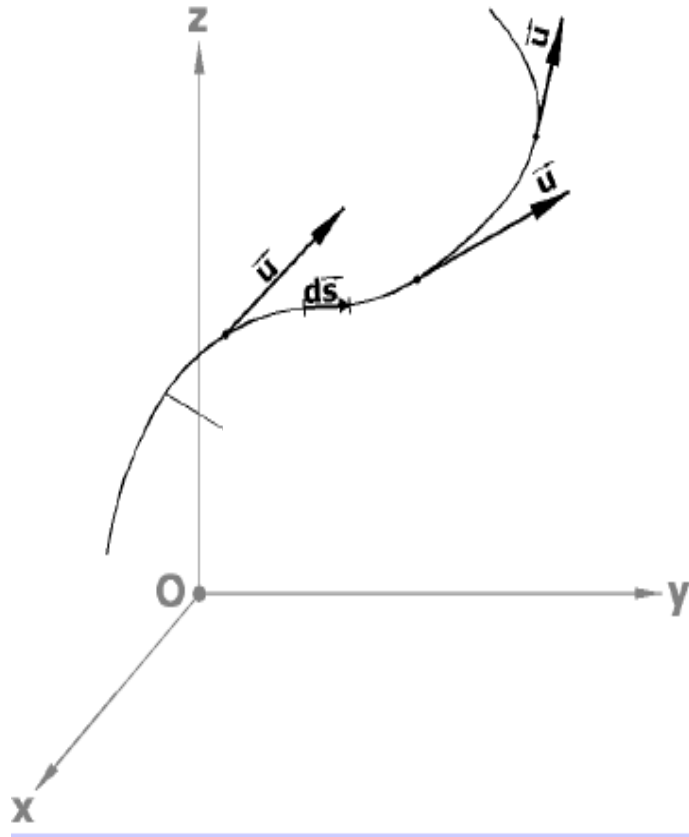
or in vector notation:

lub w zapisie wektorowym:

$$\frac{D\bar{u}}{Dt} = \frac{\partial \bar{u}}{\partial t} + \bar{u} \bullet \text{grad} \bar{u} = \frac{\partial \bar{u}}{\partial t} + (\bar{u} \nabla) \bar{u}$$

Stream line is the line of the vector field of velocity, i.e. the line tangent to the velocity vectors in every point of the velocity field in the given instant of time. If ds is the element of the stream line, and u – the velocity vector, then we have:

Linia prądu jest to linia pola wektorowego prędkości, czyli linia styczna do wektora prędkości w każdym punkcie pola w danej chwili czasu. Jeżeli ds jest elementem linii prądu, a u – wektorem prędkości, to mamy:



$$d\vec{s} \times \vec{u} = 0$$

condition of tangentially
warunek styczności

or:
czyli:

$$u_z dy - u_y dz = 0$$

$$u_x dz - u_z dx = 0$$

$$u_y dx - u_x dy = 0$$

what leads to the stream line equation:
co prowadzi do równania linii prądu:

$$\frac{dx}{u_x} = \frac{dy}{u_y} = \frac{dz}{u_z}$$

In general only one, univocally determined, stream line passes through every point of the velocity field. If more stream lines converge in one point of the field then this is a **singular point**. If we draw stream lines through a line not being a stream line, we obtain a **stream surface**. If this line is a closed curve, we obtain a **stream tube**. If this tube has an infinitesimal cross-section, we obtain a **stream filament**. Stream tube is a good model of a pipeline, for which we may determine:

Na ogół przez każdy punkt pola prędkości przechodzi jedna linia prądu dająca się wyznaczyć w sposób jednoznaczny. Jeżeli w jakimś punkcie pola zbiega się więcej linii prądu, to jest to **punkt osobliwy**. Jeżeli przez krzywą nie będącą linią prądu poprowadzimy linie prądu, to uzyskamy **powierzchnię prądu**. Jeżeli jest to krzywa zamknięta, to uzyskamy **rukę prądu**. Jeżeli przekrój tej rurki jest infinitezymalny, to uzyskamy **włókno prądu**. Rurka prądu jest dobrym modelem rurociągu, dla którego można wyznaczyć:

volumetric intensity of flow:
objętościowe natężenie przepływu:

$$Q = \int u_n dS$$

volumetric mean velocity:
objętościową prędkość średnią:

$$\tilde{u} = \frac{1}{S} \int u_n dS$$

mass intensity of flow:
masowe natężenie przepływu:

$$M = \int \rho u_n dS$$

mass mean velocity:
masową prędkość średnią:

$$\tilde{u} = \frac{\int \rho u_n dS}{\int \rho dS}$$

where: u_n
gdzie:

is the velocity component normal to the cross-section S
jest składowa prędkości normalną do przekroju rurki S

Path of the fluid element or trajectory is the geometrical location of the points in the field of flow through which the element passes in the consecutive instants of time.

Tor elementu plynulub trajektoria jest to miejsce geometryczne punktów w polu przepływu przez które przechodzi element w kolejnych chwilach czasu.

Vector equation of the path:

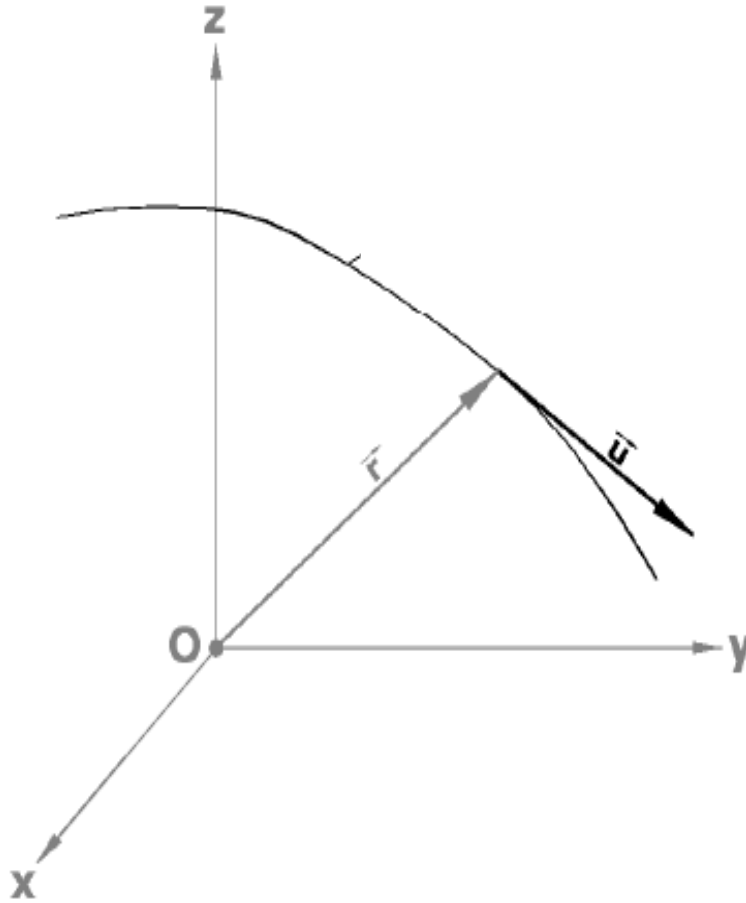
Wektorowe równanie toru:

$$\frac{d\bar{r}}{dt} = \bar{u}(\bar{r}, t)$$

In the scalar form:

W postaci skalarnej:

$$\begin{aligned} \frac{dx}{dt} &= u_x(x, y, z, t) & \frac{dy}{dt} &= u_y(x, y, z, t) \\ \frac{dz}{dt} &= u_z(x, y, z, t) \end{aligned}$$



Solution requires taking into account the initial conditions for $t = t_0$

Rozwiązanie wymaga uwzględnienia warunków początkowych dla $t = t_0$

$$x(t) = x_0 \quad y(t) = y_0 \quad z(t) = z_0$$

Flow Visualization

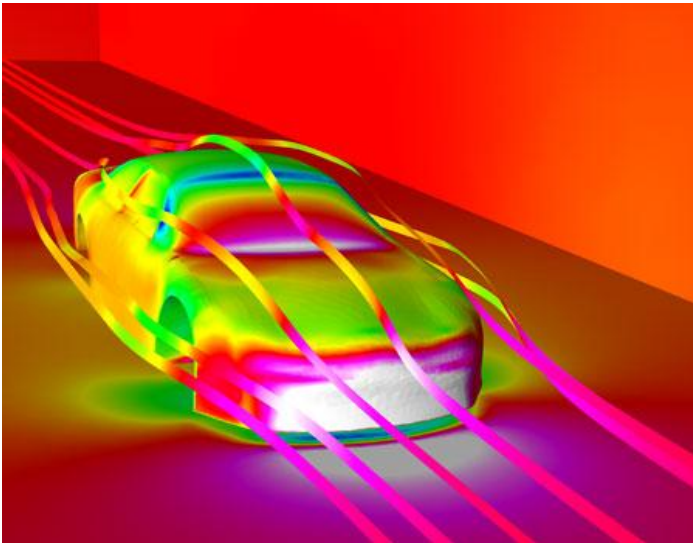
Flow visualization is the visual examination of flow-field features.

Important for both physical experiments and numerical (CFD) solutions.

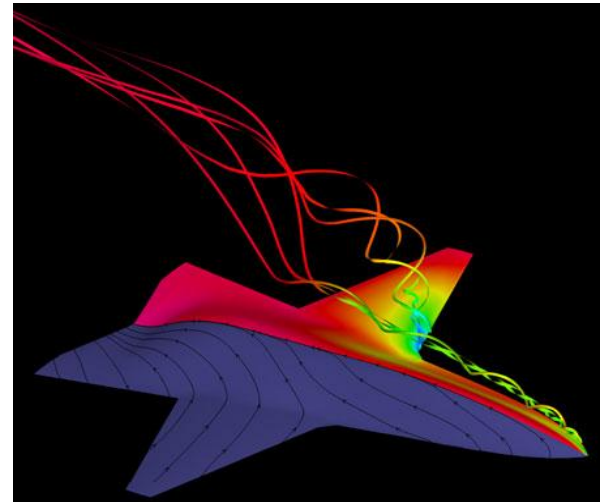
Numerous methods

- Streamlines and streamtubes
- Pathlines
- Streaklines
- Timelines
- Refractive techniques
- Surface flow techniques

NASCAR surface pressure contours
and streamlines

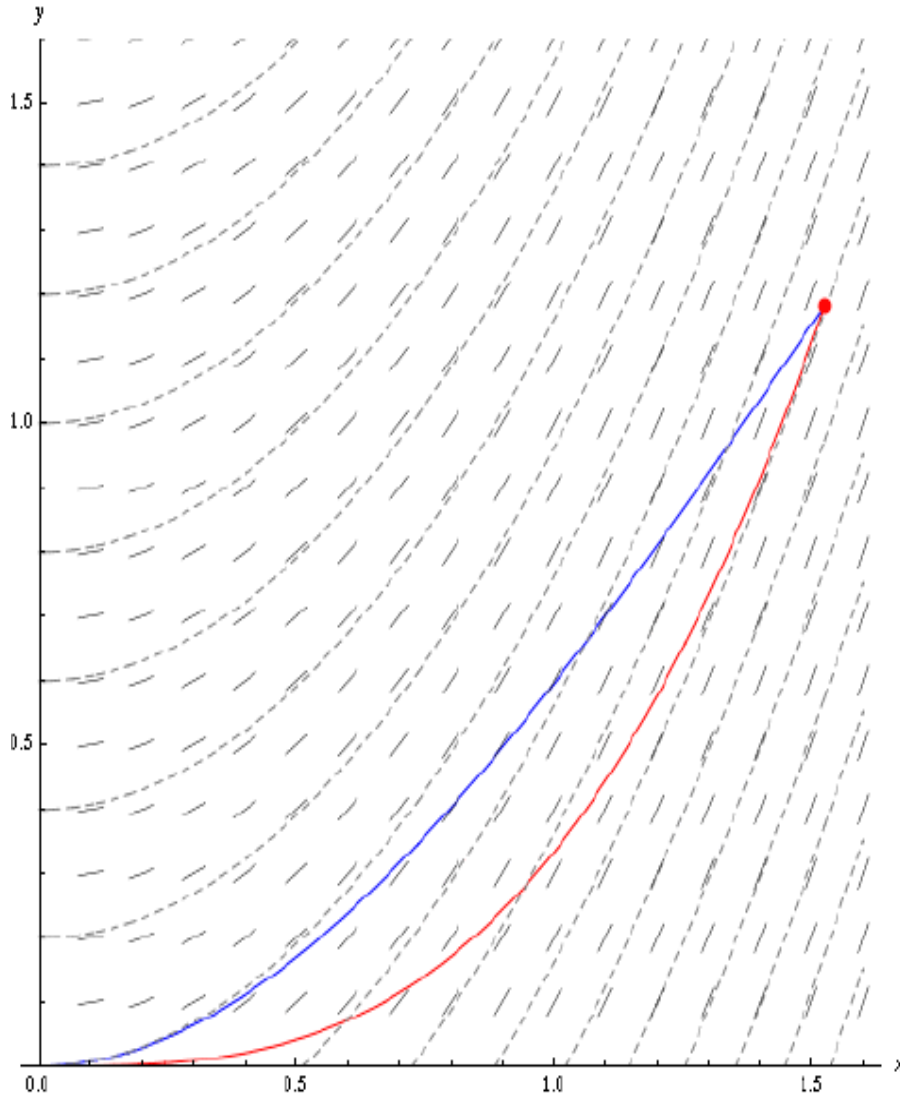


Airplane surface pressure contours,
volume streamlines, and surface
streamlines



In an unsteady flow the stream lines, paths of the elements and streak lines **do not coincide.**

W przepływie niestacjonarnym linie prądu, tory elementów płynu i linie wysnute **nie pokrywają się.**



Streamlines – grey colour

Paths of the elements – red colour

Streak lines – blue colour

Streak line is the trace of the fluid element drifting in the unsteady velocity field of the moving fluid.

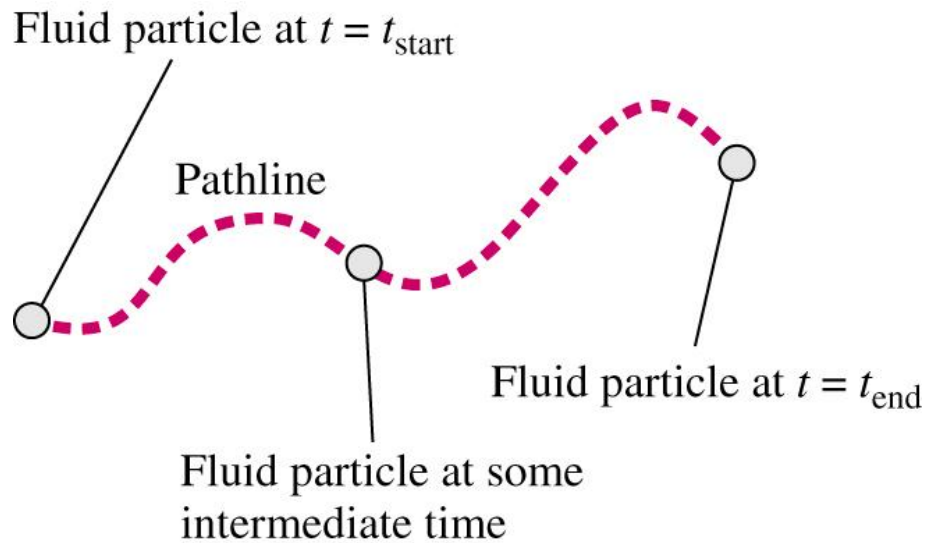
Linie prądu – kolor szary

Tor elementu – kolor czerwony

Linia wysnuta – kolor niebieski

Linia wysnuta jest to ślad ruchu elementu płynu „znoszony” przez zmieniające się pole prędkości.

Pathlines



- A **Pathline** is the actual path traveled by an individual fluid particle over some time period.
- Same as the fluid particle's material position vector

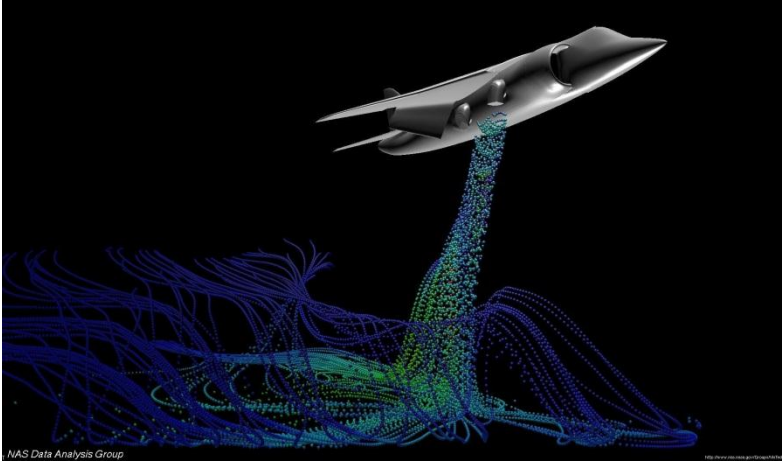
$$(x_{particle}(t), y_{particle}(t), z_{particle}(t))$$

- Particle location at time t :

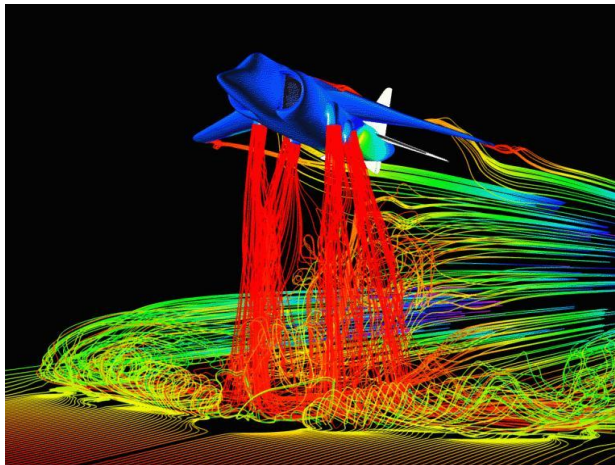
$$\vec{x} = \vec{x}_{start} + \int_{t_{start}}^t \vec{V} dt$$

- Particle Image Velocimetry (PIV) is a modern experimental technique to measure velocity field over a plane in the flow field.

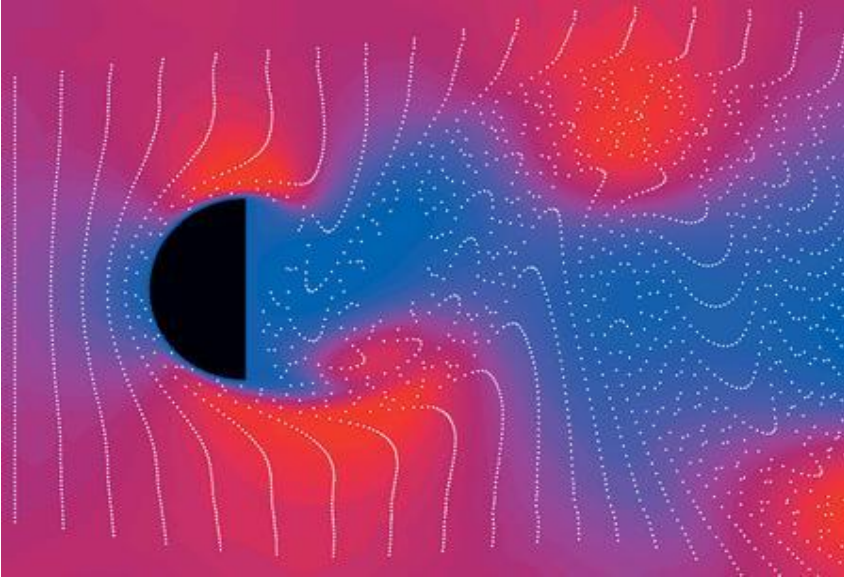
Streaklines



- A **Streakline** is the locus of fluid particles that have passed sequentially through a prescribed point in the flow.
- Easy to generate in experiments: dye in a water flow, or smoke in an airflow.



Timelines



A **Timeline** is the locus of fluid particles that have passed sequentially through a prescribed point in the flow.

Timelines can be generated using a hydrogen bubble wire.

Plots of Data

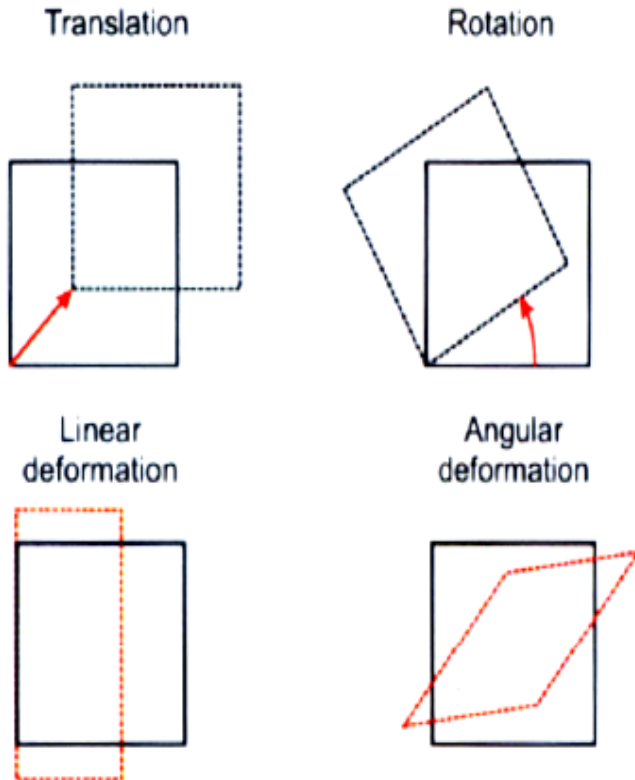
- A **Profile plot** indicates how the value of a scalar property varies along some desired direction in the flow field.
- A **Vector plot** is an array of arrows indicating the magnitude and direction of a vector property at an instant in time.
- A **Contour plot** shows curves of constant values of a scalar property for magnitude of a vector property at an instant in time.

Chapter 7 – A general motion of the fluid element

Ruch ogólny elementu płynu

A general motion of the rigid body may be considered as a sum of linear translation and rotation. As the fluids are not rigid, in their motion the deformation of the fluid elements must be additionally considered

Ruch ogólny ciała sztywnego można przedstawić jako sumę przemieszczenia liniowego i obrotu. Ponieważ płyny nie mają sztywności postaciowej, w ruchu płynu dochodzi dodatkowo do odkształcenia elementu płynu.

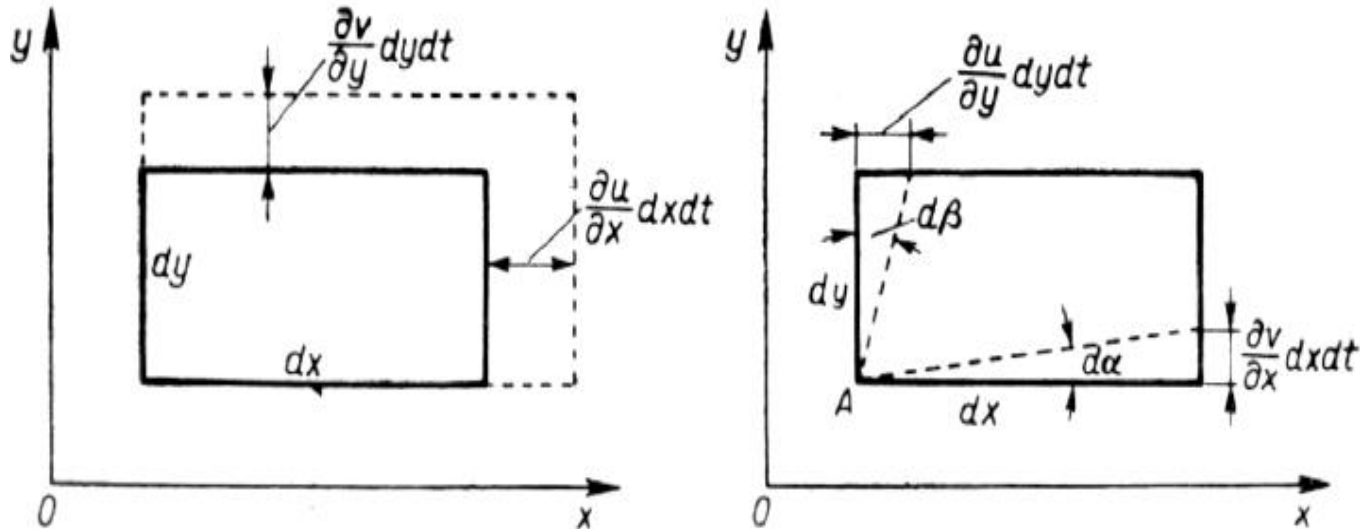


Thus the general motion of the fluid element may be treated as the super position of the linear translation, rotation around the temporary centre and deformation, which in turn may be divided into linear deformation and angular (shearing) deformation.

Ruch ogólny elementu płynu można więc traktować jako superpozycję przemieszczenia liniowego (translacji), obrotu względem chwilowego bieguna oraz odkształcenia (deformacji), które z kolei można podzielić na liniowe (objętościowe) i kątowe (postaciowe).

Deformations in the two-dimensional case

Odształcenia w przypadku dwuwymiarowym



Velocity of the fluid motion is:

$$\bar{u} = \bar{i}u + \bar{j}v$$

Prędkość ruchu płynu zapisujemy jako:

The linear deformation of the fluid element takes place when the velocity component u varies in direction x and/or the velocity component v varies in direction y (left side of the picture). This may lead to the change of the element volume in the time dt by the value:

Do odkształcenia liniowego elementu płynu dochodzi gdy składowa prędkości u zmienia się w kierunku x i/lub składowa prędkości v zmienia się w kierunku y (lewa strona rysunku). Prowadzi to do przyrostu objętości elementu w czasie dt o wartość:

$$\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) dx dy dt$$

Where the quantities in parantheses are the linear deformation velocities:

gdzie wielkości w nawiasie są prędkościami odkształcenia liniowego

$$\epsilon_{xx} = \frac{\partial u}{\partial x}$$

$$\epsilon_{yy} = \frac{\partial v}{\partial y}$$

The angular (shearing) deformation of the fluid element takes place when the velocity component u varies in the direction y and/or the velocity component v varies in the direction x (right hand side of the picture). This leads to the rotation of the element walls by the angles:

Do odkształcenia postaciowego elementu płynu dochodzi gdy składowa prędkości u zmienia się w kierunku y i/lub składowa prędkości v zmienia się kierunku x (prawa strona rysunku). Prowadzi to do obrotu ścianek elementu płynu o kąty:

$$d\alpha = \frac{\partial v}{\partial x} dt \qquad d\beta = \frac{\partial u}{\partial y} dt$$

The measure of the combined angular deformation is the expression:

Miarą prędkości łącznego odkształcenia postaciowego jest wyrażenie:

$$\varepsilon_{xy} = \frac{1}{2} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)$$

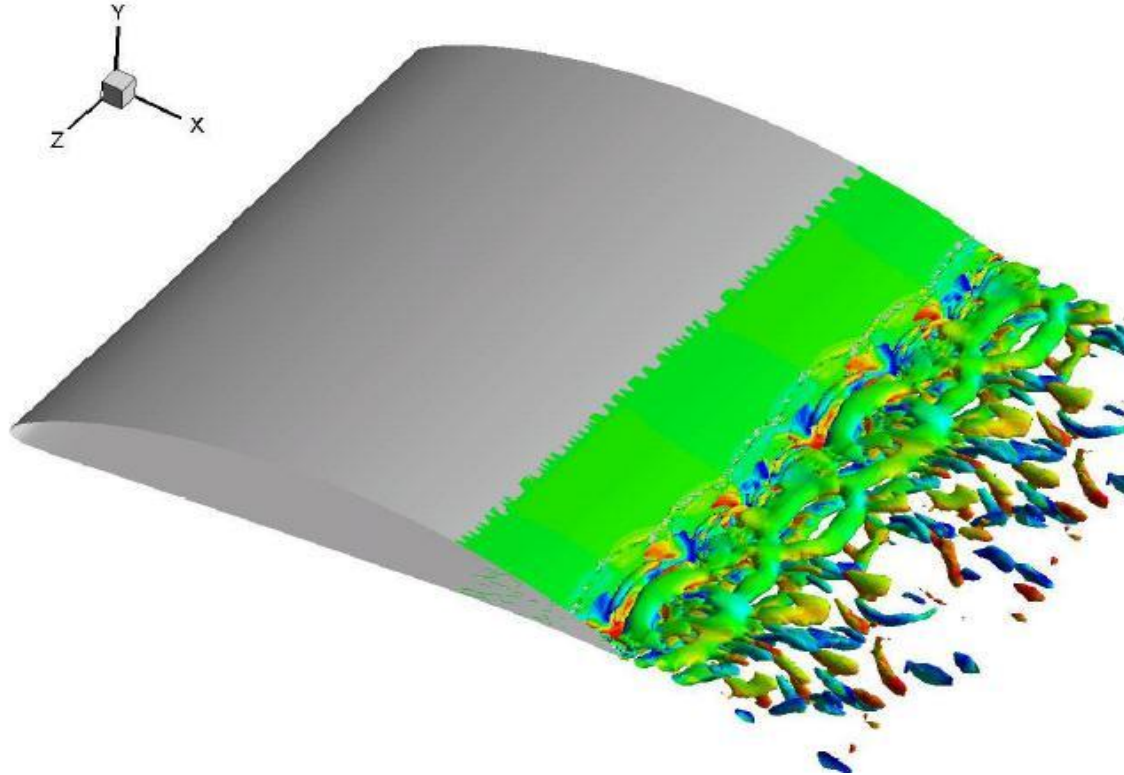
The rigid rotation of the fluid element may be regarded as the sum of two deformations selected in such a way that the angles between the element walls remain the right angles. The angular velocity of such rotation may be written as:

Sztywny obrót elementu płynu można traktować jako sumę dwóch odkształceń postaciowych tak dobranych, że kąty pomiędzy bokami elementu pozostają proste. Prędkość kątową takiego obrotu można zapisać jako:

$$\Omega_z = \frac{1}{2} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)$$

Shear Strain Rate

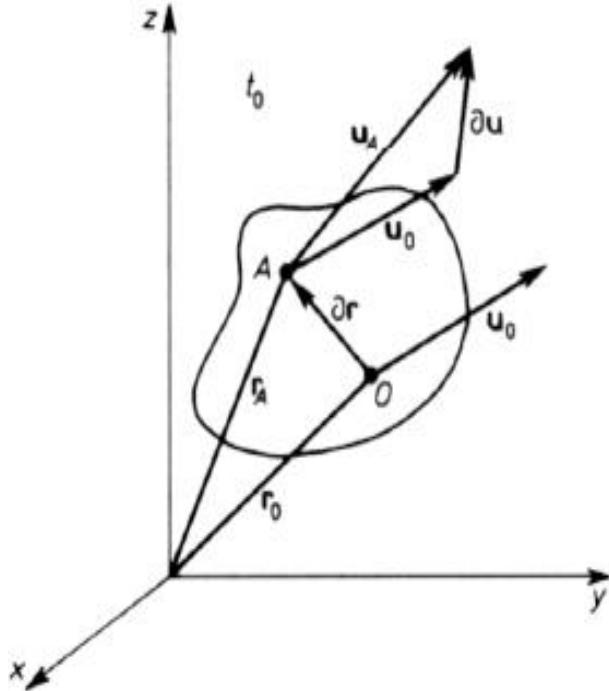
Example: Visualization of trailing-edge turbulent eddies for a hydrofoil with a beveled trailing edge



Feature extraction method is based upon eigen-analysis of the strain-rate tensor.

Deformations in the three-dimensional case

Odształcenia w przypadku trójwymiarowym



The fluid element performs an arbitrary motion composed of translation with velocity \bar{u}_0 , rotation around the centre O and deformation. Due to the rotation and deformation the vector $\partial\bar{r}$ linking point A with the centre changes. In a general case this vector change sits module and its direction. It may be written:

Element płynu wykonuje ruch ogólny złożony z translacji z prędkością \bar{u}_0 oraz obrotu względem bieguna O i deformacji. Na skutek obrotu i deformacji ulega zmianie wektor $\partial\bar{r}$ łączący punkt A z biegunem. W ogólnym przypadku wektor ten doznaje obrotu i zmiany długości. Można napisać:

$$d(\partial\bar{r}) = (\bar{u}_A - \bar{u}_0)dt$$

If the distance between the points O and A is assumed to be small, the difference of their velocities may be developed into a Taylor series and only the first term of this development may be taken into account:

Przy założeniu małej odległości pomiędzy punktami O i A można różnicę prędkości rozwinąć w szereg Taylora i wziąć uwagę tylko pierwszy wyraz:

$$\bar{u}_A = \bar{u}_0 + \nabla\bar{u}_0 \cdot (\bar{r}_A - \bar{r}_0) + \frac{1}{2} \nabla^2 \bar{u}_0 \cdot (\bar{r}_A - \bar{r}_0)^2 \dots$$

hence:

czyli:

$$\partial \bar{u} = \bar{u}_A - \bar{u}_0 = \frac{\partial(\delta \bar{r})}{\partial t} = \nabla \bar{u}_0 \cdot \partial \bar{r}$$

where: $\nabla \bar{u}_0$ Is the tensor of relative velocity of the point A with respect to the centre O

gdzie: $\nabla \bar{u}_0$ jest tensorem prędkości względnej punktu A w zględem bieguna O

$$\nabla \bar{u} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

Where the velocity vector has the form:

Gdzie wektor prędkości ma postać:

$$\bar{u} = \bar{i}u + \bar{j}v + \bar{k}w$$

The tensor of relative velocity may be considered as the sum of two tensors: anti-symmetric and symmetric. The anti-symmetric tensor describes the rotation of the fluid element as a rigid body. Its terms represent the components of the angular velocity vector ω :

Tensor prędkości względnej może być przedstawiony jako suma dwóch tensorów: antysymetrycznego i symetrycznego. Tensor antysymetryczny opisuje obrót elementu płynu jako ciała sztywnego. Jego wyrazy są składowymi prędkości kątowej obrotu ω .

$$[\Omega] = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}$$

where:

gdzie:

$$\bar{\omega} = \bar{i}\omega_x + \bar{j}\omega_y + \bar{k}\omega_z$$

$$\bar{\omega} = \frac{1}{2} \text{rot} \bar{u}$$

The respective terms of the tensor are described by the following expressions:

Poszczególne składowe tensora wyrażają się zależnościami:

$$\omega_x = \frac{1}{2} \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial x} \right)$$

$$\omega_y = \frac{1}{2} \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right)$$

$$\omega_z = \frac{1}{2} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)$$

The symmetric tensor describes the deformation of the fluid element and it is known as the rate of deformation tensor:

Tensor symetryczny opisuje deformację elementu płynu i nosi nazwę tensora prędkości deformacji:

$$[D] = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix}$$

where the respective terms are described by the expressions:

gdzie poszczególne składowe wyrażają się zależnościami:

$$\varepsilon_{xx} = \frac{\partial u}{\partial x}$$

$$\varepsilon_{xy} = \varepsilon_{yx} = \frac{1}{2} \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right)$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y}$$

$$\varepsilon_{yz} = \varepsilon_{zy} = \frac{1}{2} \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \right)$$

$$\varepsilon_{zz} = \frac{\partial w}{\partial z}$$

$$\varepsilon_{xz} = \varepsilon_{zx} = \frac{1}{2} \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right)$$

Finally, the general motion of the fluid element may be described by the following relation:

Ostatecznie ogólny ruch elementu płynu można opisać następującą zależnością:

$$\bar{u}_A = \bar{u}_0 + \bar{\omega}_0 \times \partial \bar{r} + [D]_0 \cdot \partial \bar{r}$$

The first Helmholtz theorem

The velocity of an arbitrary point of the fluid element is composed of:

- translation velocity of the point selected as the centre,
- rotation velocity around the axis passing through this centre (the vector of this velocity defines the axis of rotation),
- deformation velocity of the fluid element.

In comparison with the analogical motion of the rigid body the following differences may be noticed:

- the formula for the fluid is valid only close to the rotation centre,
- in the fluid there is an additional velocity of deformation.

Pierwsze twierdzenie Helmholtza

W porównaniu z analogicznym ruchem ciała sztywnego można stwierdzić następujące różnice:

- wzór dla płynu jest ważny tylko w bliskim otoczeniu bieguna
- w płynie dodatkowo występuje prędkość deformacji

Prędkość dowolnego punktu elementu płynu składa się:

- prędkości postępowej punktu obranego za biegun-prędkości obrotowej wokół osi przechodzącej przez biegun (wektor tej prędkości wyznacza oś obrotu)-prędkości deformacji elementu płynu.

Rotational motion of the fluid

Ruch wirowy płynu

The flow is rotational if every where or almost every where (except the finite number of points, lines or surfaces) the rotation of the velocity field is not equal to zero. Then the vector of rotation (or vorticity vector) may be ascribed to every or almost every point in space:

Wirowym nazywamy przepływ, w którym wszędzie lub prawie wszędzie (czyli z wyjątkiem skończonej liczby punktów, linii i powierzchni) rotacja pola prędkości jest różna od zera. Wtedy każdemu lub prawie każdemu punktowi przestrzeni można przypisać wektor wirowości:

$$\bar{\Omega} = \text{rot} \bar{u} = 2\bar{\omega}$$

The components of the vorticity vector are defined as:

Składowe wektora wirowości wyrażają się zależnościami:

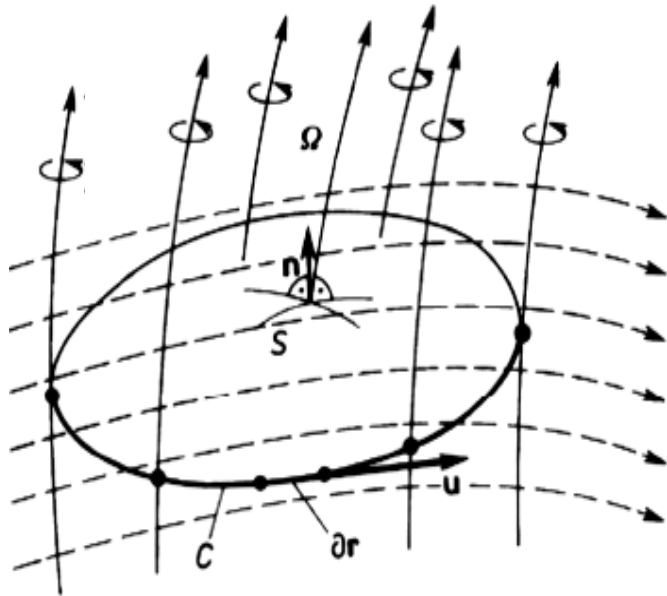
$$\Omega_x = 2\omega_x = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}$$

$$\Omega_y = 2\omega_y = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}$$

$$\Omega_z = 2\omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

Through the analogy to the streamlines the vortex lines may be determined as the lines of vector field of vorticity, i.e. the lines tangent to the vorticity vectors in every point of space.

Przez analogię do linii prądu można określić linie wirowe jako linie pola wektorowego wirowości, czyli linie styczne w każdym punkcie pola do wektorów wirowości.



Equation of the vortex line:

Równanie linii wirowej:

$$\frac{dx}{\Omega_x} = \frac{dy}{\Omega_y} = \frac{dz}{\Omega_z}$$

Circulation of the velocity vector is defined as:

Cyrkulacja wektora prędkości jest definiowana jako:

$$\Gamma = \oint_C \bar{u} \cdot d\bar{r} = \oint_C (u dx + v dy + w dz) = \int_S \text{rot} \bar{u} \cdot dS$$

Stokes theorem: Circulation of the velocity around an arbitrary contour C is equal to the stream of vorticity passing through an arbitrary surface based on this contour.

Twierdzenie Stokesa: cyrkulacja prędkości wzdłuż dowolnego konturu C jest równa strumieniowi wirowości przez dowolną powierzchnię objętą konturem.

Vortex lines passing through a line which is not a vortex line form the **vortex surface**. If this line is a closed curve, they form a **vortex tube**. The vortex tube of an infinitesimal diameter is a **vortex filament**.

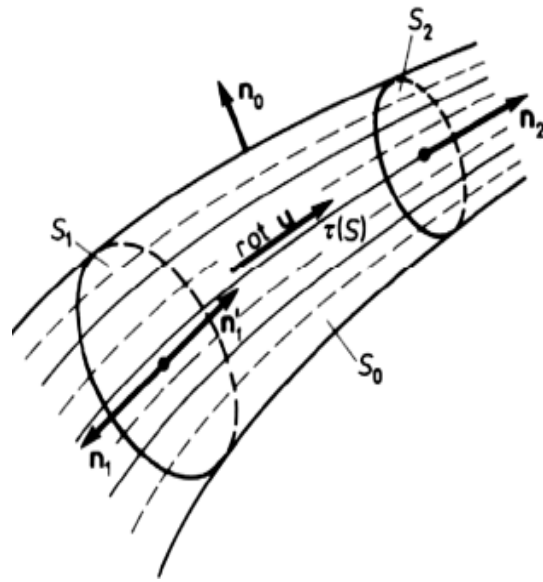
Linie wirowe przechodzące przez krzywą będącą linią wirową tworzą **powierzchnię wirową**. Jeżeli ta krzywa jest krzywą zamkniętą to powstaje **rurka wirowa**. Rurka wirowa o infinitezymalnej średnicy to **włókno wirowe**.

Thomson theorem: in the flow of an ideal barotropic fluid under the action of the potential field of mass forces the circulation of velocity around an arbitrary closed fluid contour does not change with time.

Twierdzenie Thomsona: w przepływie idealnego płynu barotropowego znajdującego się pod działaniem potencjalnego pola sił masowych cyrkulacja prędkości wzdłuż dowolnej zamkniętej linii płynnej nie zmienia się w czasie

Second theorem of Helmholtz: in the flow of an ideal barotropic fluid under the action of the potential field of mass forces the intensity of the vortex filament does not change along its length and it is constant in time.

Drugie twierdzenie Helmholtza: w przepływie idealnego płynu barotropowego z znajdującego się pod działaniem potencjalnego pola sił masowych natężenie włókna wirowego nie zmienia się wzdłuż jego długości i jest stałe w czasie.



Conclusions:

- a vortex filament may not disappear or be generated in the fluid,
- a vortex filament may form a closed curve,
- a vortex filament may terminate at the free surface or at the rigid body surface,
- the same fluid elements participate in vortex motion at all times.

Wnioski:

- włókno wirowe nie może zanikać ani powstawać w płynie,
- włókno wirowe może tworzyć krzywą zamkniętą,
- włókno wirowe może się kończyć na swobodnej powierzchni lub na ścianach sztywnych,
- w ruchu wirowym biorą udział te same elementy płynu.

In practical modeling the flow may be divided into the rotational flow region and irrotational flow region. Both these regions are mutually interdependent. The rotational region may be modeled by vortex filaments. In such a case it is important to determine the velocity field generated by the vorticity field, i.e. the operation opposite to the calculation of rotation of the velocity field.

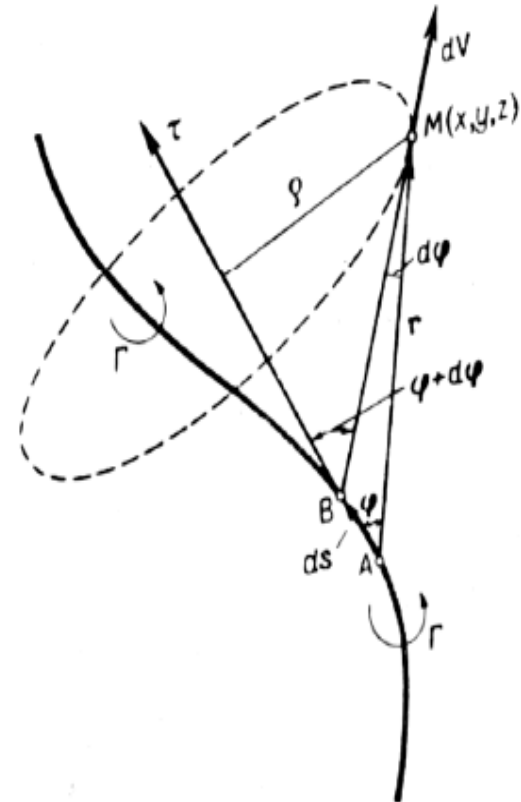
W praktycznym modelowaniu przepływu można podzielić obszar o ruchu wirowym i obszar o ruchu bezwirowym. Oba te obszary są wzajemnie współzależne. Obszar o ruchu wirowym może być modelowany włóknami wirowymi. Istotne staje się wtedy wyznaczanie pola prędkości generowanego przez pole wirowości, czyli operacja odwrotna do obliczania rotacji pola prędkości.

The Biot-Savart formula

Wzór Biota-Savarta

$$d\vec{V} = \frac{\Gamma}{4\pi} \frac{d\vec{s} \times \vec{r}}{r^3}$$

$$\vec{V} = \frac{\Gamma}{4\pi} \int_L \frac{d\vec{s} \times \vec{r}}{r^3}$$



Vorticity and Rotationality

