Lecture No. 5 -

Chapter 8 – Mass conservation equation

Równanie zachowania masy

Principle of mass conservation: in a closed physical system mass cannot be generated or annihilated.

Prawo zachowania masy: w zamkniętym układzie fizycznym masa nie może powstaćani nie może ulec anihilacji.

Assumptions:

- we consider an unsteady three-dimensional flow of a compressible fluid,
- the fluid fills the space in a continuous way (no bubble setc.),
- we apply the Eulerian approach – a stationary control volume surrounded by a control surface. With these assumptions the mass conservation principle reads: the change of mass in the volume= the flow of mass through the surface

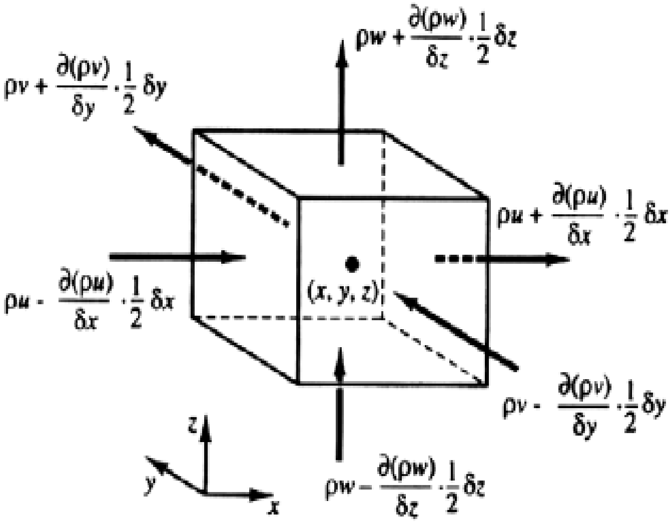
The change of mass in the control volume is:

Założenia:

- rozpatrujemy nieustalony trójwymiarowy przepływ płynu ściśliwego,
- płyn w całości wypełnia przestrzeń (brak nieciągłości, pęcherzy itp.),
- stosujemy opis Eulera –nieruchoma objętośćkontrolna ograniczona powierzchnią kontrolną. Przy tych założeniach prawo zachowania masy brzmi: przyrost masy w objętości = przepływ masy przez powierzchnię

Przyrost masy w objętości kontrolnej wynosi:

$$\frac{\partial}{\partial t}(\rho \delta x \delta y \delta z) = \frac{\partial \rho}{\partial t} \delta x \delta y \delta z$$



In turn the flow through the control surface is:

Z kolei przepływ przez powierzchnię kontrolną wynosi:

$$\left(\rho u-\frac{\partial(\rho u)}{\partial x}\frac{1}{2}\delta x\right)\delta y\delta z-\left(\rho u+\frac{\partial(\rho u)}{\partial x}\frac{1}{2}\delta x\right)\delta y\delta z+$$

$$+\left(\rho v-\frac{\partial(\rho v)}{\partial y}\frac{1}{2}\delta y\right)\delta x\delta z-\left(\rho v+\frac{\partial(\rho v)}{\partial y}\frac{1}{2}\delta y\right)\delta x\delta z+$$

$$+\left(\rho w-\frac{\partial(\rho w)}{\partial z}\frac{1}{2}\delta z\right)\delta x\delta y-\left(\rho w+\frac{\partial(\rho w)}{\partial z}\frac{1}{2}\delta z\right)\delta x\delta y$$

Equating of both above expressions leads to (after dividing both sides by the control volume):

Porównanie obu wyrażeń i podzielenie stronami przez objętość kontrolną prowadzi do wzoru:

$$\frac{\partial \rho}{\partial t}+\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}=\frac{\partial \rho}{\partial t}+div(\rho\bar{u})=0$$

In the case of the steady flow of the compressible fluid the mass conservation equation takes the form:

W przypadku ustalonego przepływu płynu ściśliwegorównanie zachowania masy przybiera postać:

$$\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}=div(\rho\bar{u})=0$$

In the case of the steady flow of the incompressible fluid the mass conservation equation takes the form:

W przypadku ustalonego przepływu płynu nieściśliwegorównanie zachowania masy przybiera postać:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=div\bar{u}=0$$

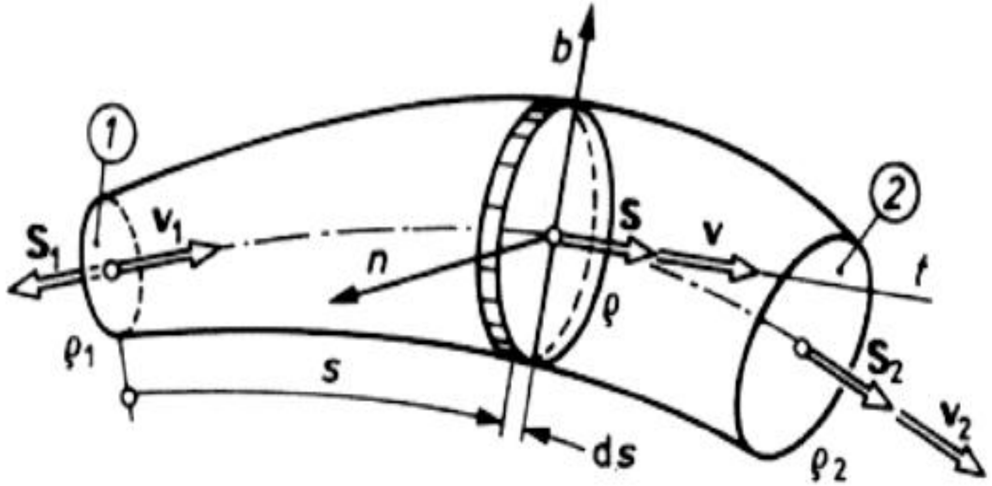
In the case of the moving fluid element (Lagrangian approach) the mass conservation equation takes the form:

W przypadku poruszającego się elementu płynu (opis Lagrange'a) równanie zachowania masy przybiera postać:

$$\frac{\partial \rho}{\partial t}+div(\rho\bar{u})=\frac{\partial \rho}{\partial t}+\bar{u}\cdot grad\rho+\rho div\bar{u}=\frac{D\rho}{Dt}+\rho div\bar{u}$$

Mass conservation equation for a stream tube

Równanie zachowania masy dla rurki prądu



The difference of mass flow between the cross-sections 1 and 2:

Różnica natężeń przepływu masy pomiędzy przekrojami 1 i 2:

$$\int_{S_1} \rho(s,t) u(s,t) dS - \int_{S_2} \rho(s,t) u(s,t) dS = \int_2^1 \frac{\partial}{\partial s}\left(\tilde{\rho}\tilde{u}S\right) ds$$

The corresponding average values are defined as:

Odpowiednie wartości średnie są zdefiniowane następująco:

$$\tilde{\rho}_1 = \frac{1}{S_1}\int_{S_1} \rho_1 dS_1 \qquad \tilde{\rho}_2 = \frac{1}{S_2}\int_{S_2} \rho_2 dS_2$$

$$\tilde{u}_1 = \frac{1}{S_1\tilde{\rho}_1}\int_{S_1} \rho_1 u_1 dS_1 \qquad \tilde{u}_2 = \frac{1}{S_2\tilde{\rho}_2}\int_{S_2} \rho_2 u_2 dS_2$$

This difference may be generated due to the compression or expansion of the fluid in the control volume limited by the cross-sections 1 and 2. In such a case the rate of change of mass in the control volume is:

Różnica ta może powstaćw wyniku kompresji lub ekspansji płynu w obszarze kontrolnym ograniczonym przekrojami 1 i 2. W takim przypadku masa w obszarze kontrolnym zmienia sięz prędkością:

$$\frac{\partial}{\partial t}\int_1^2 \tilde{\rho}(s,t)S(s,t)ds$$

According to the mass conservation principle the rate of change of mass in the control volume must be equal to the flow of mass through the control surface.

Zgodnie z zasadązachowania masy zmiana masy w objętości musi byćrówna strumieniowi masy przez powierzchnięograniczającą.

$$\frac{\partial}{\partial t}\int_1^2 \tilde{\rho}S ds = -\int_1^2 \frac{\partial(\tilde{\rho}\tilde{u}S)}{\partial s}ds$$

or:

czyli:

$$\int_1^2 \left[\frac{\partial(\tilde{\rho}S)}{\partial t} + \frac{\partial(\tilde{\rho}\tilde{u}S)}{\partial s}\right]ds = 0$$

As the above equation is valid for an arbitrarily selected cross-sections 1 and 2, the function under the integral should be equal to zero. This leads to the following mass conservation equation:

Ponieważpowyższa równośćzachodzi dla dowolnie wybranych przekrojów 1 i 2, to funkcja podcałkowa powinna byćrówna zero. Prowadzi to do następującego równania zachowania masy:

$$\frac{\partial(\tilde{\rho}S)}{\partial t}+\frac{\partial(\tilde{\rho}\tilde{u}S)}{\partial s}=0$$

For an incompressible fluid (constant density) we obtain:

Dla płynu nieściśliwego (o stałej gęstości) otrzymujemy:

$$\frac{\partial S}{\partial t}+\frac{\partial(\tilde{u}S)}{\partial s}=0$$

In turn, for a steady flow we obtain:

Z kolei dla przepływu ustalonego otrzymujemy:

$$\frac{\partial(\tilde{\rho}\tilde{u}S)}{\partial s}=0$$

or:

czyli:

$$\tilde{\rho}\tilde{u}S = const$$

Conclusions:

- in the steady flow of compressible fluid the mass intensity of flow (the stream of mass of the fluid) through any cross-section of a stream tube is constant,
- in the steady flow of an incompressible fluid the volumetric intensity of flow (the stream of volume of the fluid) is constant and velocity is inversely proportional to the area of the stream tube cross-section.

Wnioski:

- w ustalonym przepływie płynu ściśliwego natężenie przepływu masy (strumień masy) przez każdy poprzeczny przekrój rurki prądu jest stałe,
- w ustalonym przepływie płynu nieściśliwego objętościowe natężenie przepływu (strumieńobjętości) jest stałe, czyli prędkośćprzepływu jest odwrotnie proporcjonalna do pola przekroju rurki prądu.

Chapter 9 – Momentum conservation equation 1

Równanie zachowania pędu 1

The second law of Newton: the rate of change of momentum of a fluid element is equal to the sum of forces acting on this element.

Druga zasada dynamiki Newtona: prędkość przyrostu pędu elementu płynu jest równa sumie sił zewnętrznych działających na ten element

$$\frac{D(m\bar{u})}{Dt} = \sum \bar{F}$$

The rate of change of momentum of the fluid element (the left hand side) is defined by the material derivative of its velocity:

Prędkość przyrostu pędu elementu płynu (czyli lewa strona równania) jest określona poprzez pochodną materialną jego prędkości:

$$\rho\frac{Du}{Dt} = \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = \frac{\partial(\rho u)}{\partial t} + div(\rho u\bar{u})$$

$$\rho\frac{Dv}{Dt} = \rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = \frac{\partial(\rho v)}{\partial t} + div(\rho v\bar{u})$$

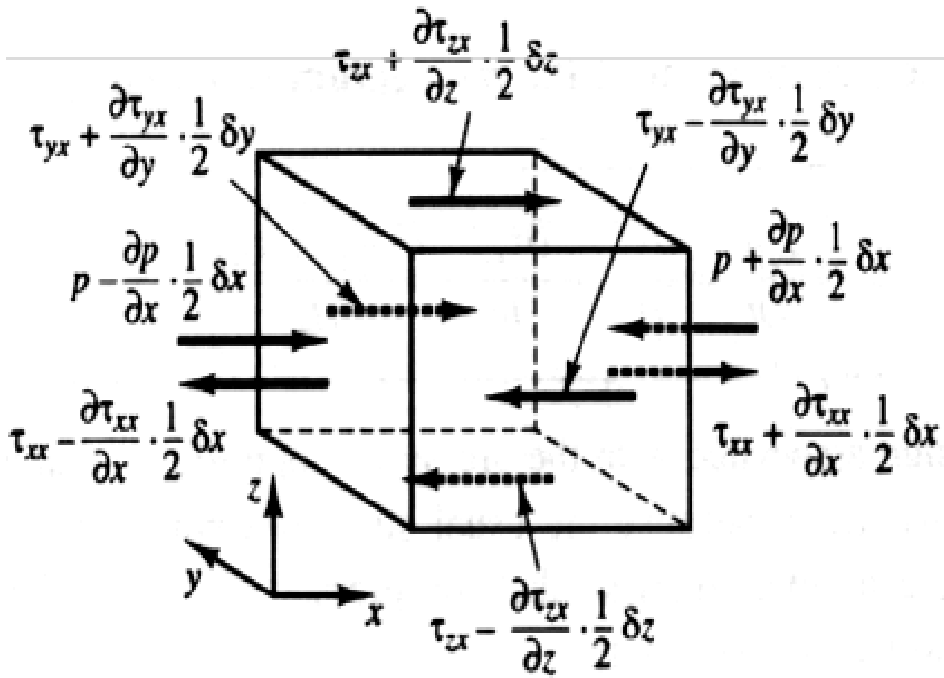
$$\rho\frac{Dw}{Dt} = \rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = \frac{\partial(\rho w)}{\partial t} + div(\rho w\bar{u})$$

The right hand side is composed of the two categories of forces:
- surface forces (pressure forces and viscosity forces),
- mass forces (gravity forces, Coriolis forces, electromagnetic forces)

For example we will formulate the complete equation for the x direction, using the system of surface forces as in the picture:

Prawą stronę równania tworzą dwie kategorie sił:
-siły powierzchniowe (siły ciśnienia i siły lepkości),
-siły masowe (siły ciężkości, siły Coriolisa, siły elektromagnetyczne)

Dla przykładu utworzymy kompletne równanie dla kierunku x, posługując się układem sił powierzchniowych jak na rysunku:



Forces acting on the element walls perpendicular to the x direction

Siły działające na ścianki elementu prostopadłe do kierunku x

$$\left[\left(p-\frac{\partial p}{\partial x}\frac{1}{2}\delta x\right)-\left(\tau_{xx}-\frac{\partial \tau_{xx}}{\partial x}\frac{1}{2}\delta x\right)\right]\delta y\delta z+\left[-\left(p+\frac{\partial p}{\partial x}\frac{1}{2}\delta x\right)+\left(\tau_{xx}+\frac{\partial \tau_{xx}}{\partial x}\frac{1}{2}\delta x\right)\right]\delta y\delta z=$$

$$=\left(-\frac{\partial p}{\partial x}+\frac{\partial \tau_{xx}}{\partial x}\right)\delta x\delta y\partial z$$

Forces acting on the element walls perpendicular to the y direction

Siły działające na ścianki elementu prostopadłe do kierunku y

$$-\left(\tau_{yx}-\frac{\partial \tau_{yx}}{\partial y}\frac{1}{2}\delta y\right)\delta x\delta z+\left(\tau_{yx}+\frac{\partial \tau_{yx}}{\partial y}\frac{1}{2}\delta y\right)\delta x\delta z=\frac{\partial \tau_{yx}}{\partial y}\delta x\delta y\delta z$$

Forces acting on the element walls perpendicular to the z direction

Siły działające na ścianki elementu prostopadłe do kierunku z

$$-\left(\tau_{zx}-\frac{\partial \tau_{zx}}{\partial z}\frac{1}{2}\delta z\right)\delta x\delta y+\left(\tau_{zx}+\frac{\partial \tau_{zx}}{\partial z}\frac{1}{2}\delta z\right)\delta x\delta y=\frac{\partial \tau_{zx}}{\partial z}\delta x\delta y\delta z$$

After adding the above expressions together and dividing by the element volume we obtain the surface forces acting in direction x:

Po dodaniu powyższych wyrażeń i podzieleniu stronami przez objętość elementu otrzymujemy siły powierzchniowe na kierunku x

$$\frac{\partial(-p+\tau_{xx})}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}+\frac{\partial \tau_{zx}}{\partial z}$$

After supplementing the expression with the unit mass force *f* and substituting it to the initial formula we obtain:

Po uzupełnieniu o składową jednostkowej siły masowej *f* i podstawieniu do wyjściowej zależności otrzymujemy:

$$\rho\frac{Du}{Dt}=\rho f_x+\frac{\partial(-p+\tau_{xx})}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}+\frac{\partial \tau_{zx}}{\partial z}$$

and analogically for the remaining directions:

i analogicznie dla pozostałych kierunków:

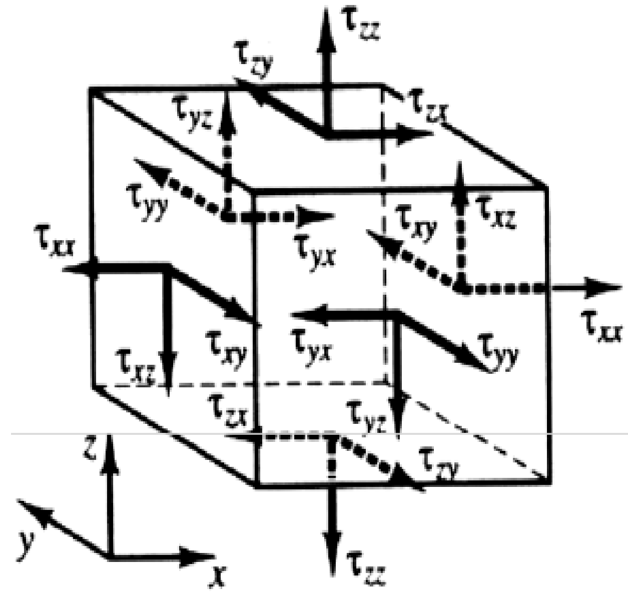
$$\rho\frac{Dv}{Dt}=\rho f_y+\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial(-p+\tau_{yy})}{\partial y}+\frac{\partial \tau_{zy}}{\partial z}$$

$$\rho\frac{Dw}{Dt}=\rho f_z+\frac{\partial \tau_{xz}}{\partial x}+\frac{\partial \tau_{yz}}{\partial y}+\frac{\partial(-p+\tau_{zz})}{\partial z}$$

Three above scalar equations may be written in the form of the equivalent single vector equation:

Trzy powyższe równania skalarne mogą być zapisane w postaci jednego równoważnego równania wektorowego:

$$\rho \frac{D\bar{u}}{Dt} = \rho\bar{f} + div[P]$$



where: $[P]$ is the tensor describing the state of stress in the fluid, as shown in the picture, supplemented with pressure added to the normal stresses

jest tensorem opisującym stan naprężenia w płynie pokazany na rysunku, poszerzony o ciśnienie dodane do naprężeń normalnych

The unknowns in the momentum conservation equations are: pressure, velocity components and stresses representing the surface viscous forces. Even after adding the mass conservation equation to the system, the number of unknowns is greater than the number of equations. In order to reduce the number of unknowns and close the system an appropriate fluid model must be introduced.

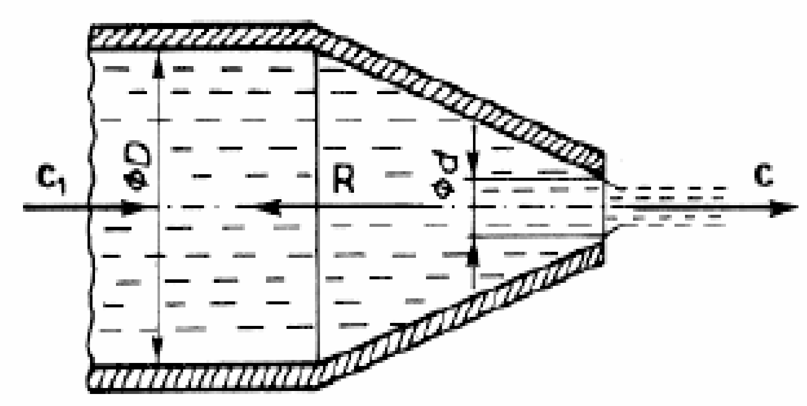
Niewiadomymi w równaniach zachowania pędu są: ciśnienie, składowe prędkości oraz naprężenia reprezentujące powierzchniowe siły lepkości. Nawet po dołączeniu równania zachowania masy liczba niewiadomych znacznie przekracza liczbę równań. Aby zredukować liczbę niewiadomych konieczne jest wprowadzenie odpowiedniego modelu płynu.

Examples of application of the momentum conservation principle to the solution of simple fluid mechanics problems

Przykłady zastosowania zasady zachowania pędu do rozwiązywania prostych zadań z mechaniki płynów

Example no. 1

Przykład 1



Water is ejected with the mean velocity c=15 [m/s] from the nozzle of diameters D=80 [mm] i d=20 [mm] . Disregarding the pressure difference calculate the hydrodynamic reaction force exerted by the water stream on the duct.
The reaction R in steady motion is:

Z dyszy o średnicach D=80 [mm] i d=20 [mm] wypływa woda ze średnią prędkością c=15 [m/s]. Pomijając różnicę ciśnień obliczyć reakcję hydrodynamiczną wywieraną przez strumień wody na dyszę.
Reakcja R w ruchu ustalonym wynosi:

$$R = \rho \cdot Q \cdot \left(c - c_1\right)$$

The flow intensity Q and velocity c1 is calculated from the continuity equation:

Natężenie przepływu Q oraz prędkość c1 obliczamy z równania iągłości:

$$Q = c \cdot \frac{\pi \cdot d^2}{4} = c_1 \cdot \frac{\pi \cdot D^2}{4}$$

Hence we have:

Wobec tego mamy:

$$Q = c \cdot \frac{\pi \cdot d^2}{4} \quad c_1 = c \cdot \frac{d^2}{D^2} \quad R = \rho \cdot c^2 \cdot \frac{\pi \cdot d^2}{4} \cdot \left(1 - \frac{d^2}{D^2}\right)$$

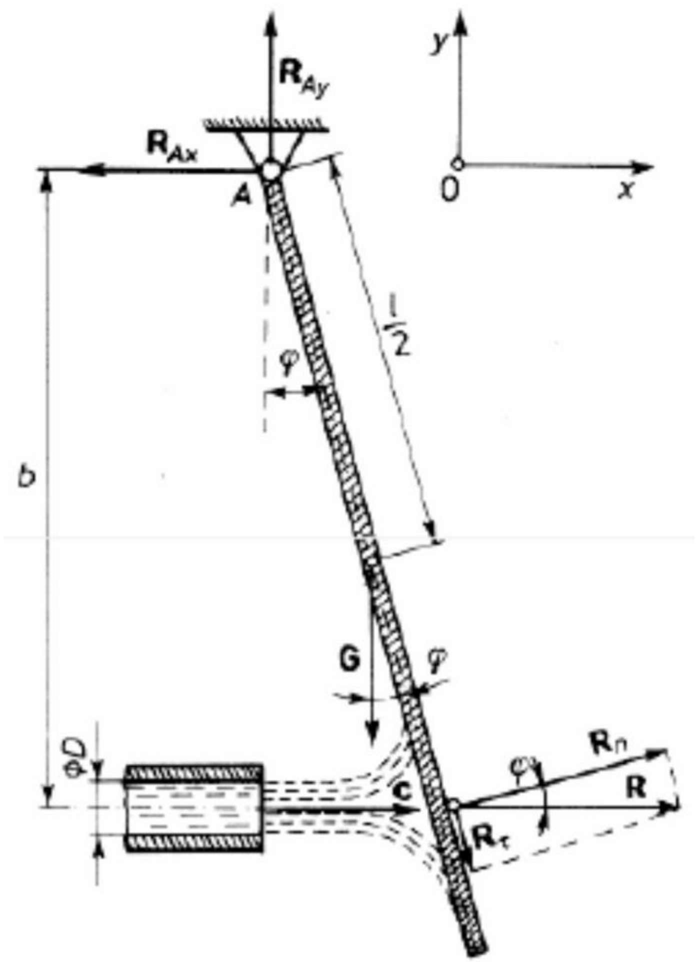
After substituting the numerical values we obtain:

Po wstawieniu wartości liczbowych otrzymujemy:

$$R = 1000 \cdot 15^2 \cdot \frac{\pi \cdot 0.02^2}{4} \cdot \left(1 - \frac{0.02^2}{0.08^2}\right) = 66.25[N]$$

Example no. 2

Przykład 2



The stream of perfect liquid of density ρ flows out of the nozzle and hits the ideally smooth plate of weight G and length l. The plate can rotate around the bearing A located at the distance b from the nozzle axis. Knowing that the intensity of the outflowing stream is Q, and the nozzle diameter is D, calculate the components of the reaction in the bearing and the angle φ of inclination of the plate in the state of equilibrium.

Strumień cieczy doskonałej o gęstości ρ wypływa z dyszy i uderza w idealnie gładką płytę o ciężarze G i długości l. Płyta może obracać się wokół łożyska A oddalonego o b od osi dyszy. Wiedząc, że natężenie wypływającego strumienia wynosi Q, a średnica dyszy D, wyznaczyć składowe reakcji w łożysku oraz kąt φ o jaki wychyli się płyta w stanie równowagi.

The hydrodynamic force R is resolved into the components normal and tangential to the plate:

Napór hydrodynamiczny R rozkładamy na składową normalną i składową styczną do płaszczyzny płyty:

$$\vec{R} = \vec{R}_n + \vec{R}_\tau$$

In the perfect liquid the tangential component is equal zero, consequently the entire reaction is represented by the normal component:

W cieczy doskonałej składowa styczna jest równa zero, wobec czego całkowity napór reprezentuje tylko składowa normalna:

$$R_n = R \cdot \cos\varphi$$

Then we have:

Dalej mamy:

$$R = \rho \cdot c \cdot Q$$

$$c = \frac{4 \cdot Q}{\pi \cdot D^2} \qquad R_n = \frac{4 \cdot \rho \cdot Q^2}{\pi \cdot D^2} \cdot \cos\varphi$$

The components of the reaction in the bearing are calculated from the equations of forces projections onto the axes x and y:

Składowe reakcji w łożysku wyznaczamy z równań rzutów sił na osie x i y:

$$\sum P_{ix} = R_n \cdot \cos\varphi - R_{Ax} = 0$$

$$\sum P_{iy} = R_{Ay} - G - R_n \cdot \sin\varphi = 0$$

Hence we obtain:

Skąd otrzymujemy:

$$R_{Ax} = \frac{4 \cdot \rho \cdot Q^2}{\pi \cdot D^2} \cdot \cos^2\varphi$$

$$R_{Ay} = G + \frac{4 \cdot \rho \cdot Q^2}{\pi \cdot D^2} \cdot \cos\varphi \cdot \sin\varphi = G + \frac{2 \cdot \rho \cdot Q^2}{\pi \cdot D^2} \cdot \sin 2\varphi$$

The inclination angle of the plate in the state of equilibrium is calculated from the equation of moments with respect to A:

Kąt nachylenia płyty w stanie równowagi wyznaczamy z równania momentów względem punktu A:

$$\sum M_A = R_n \cdot \frac{b}{\cos\varphi} - G \cdot \frac{l}{2} \cdot \sin\varphi = 0$$

We obtain:

Otrzymujemy:

$$\sin\varphi = \frac{2 \cdot R_n \cdot b}{G \cdot l \cdot \cos\varphi}$$

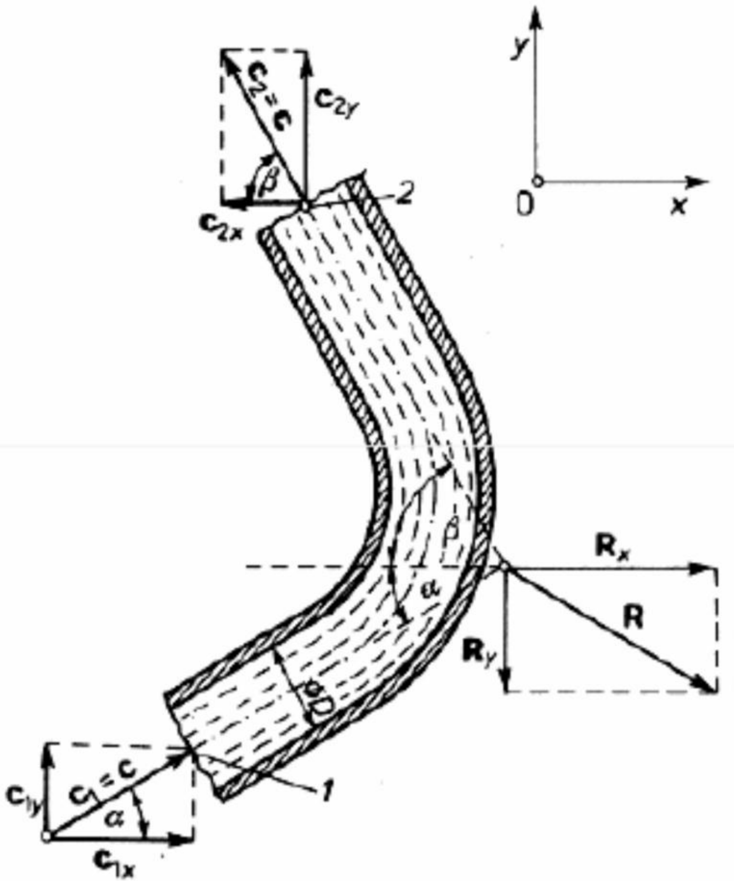
After substituting the formula for the reaction finally we get:

Po podstawieniu zależności na reakcję mamy ostatecznie:

$$\varphi = \arcsin \frac{8 \cdot \rho \cdot Q^2 \cdot b}{\pi \cdot G \cdot l \cdot D^2}$$

Example no. 3

Przykład 3



Water flows through the bent pipe of diameter D=80 [mm] with flow intensity Q=0.08 [m**3/s]. Disregarding the losses, calculate the force exerted by the water stream on the pipe. The inlet part of the pipe is located at the angle $\alpha=\pi/6$ to the horizon, while the outlet part is located at the angle $\pi/3$. In both inlet and outlet crosssections of the pipe the pressure is equal to the ambient pressure p_b.

Przez krzywak o średnicy D=80 [mm] przepływa woda z natężeniem Q=0,08 [m**3/s]. Pomijając straty obliczyć napór strumienia wody na krzywak. Część dopływowa krzywaka usytuowana jest pod kątem $\alpha=\pi/6$ do poziomu, a część odpływowa pod kątem $\pi/3$. W przekroju dopływowym i odpływowym panuje jednakowe ciśnienie otoczenia p_b.

The components of the hydrodynamic force are respectively:

Składowe naporu hydrodynamicznego wynoszą odpowiednio:

$$R_x = \rho \cdot Q \cdot \left(c_{1x} - c_{2x}\right)$$

$$R_y = \rho \cdot Q \cdot \left(c_{1y} - c_{2y}\right)$$

Where:

Gdzie:

$$c_{1x} = c \cdot \cos\alpha \qquad c_{2x} = -c \cdot \cos\beta$$

$$c_{1y} = c \cdot \sin\alpha \qquad c_{2y} = c \cdot \sin\beta$$

What gives:

Co daje:

$$R_x = \rho \cdot Q \cdot c \cdot \left(\cos\alpha + \cos\beta\right)$$

$$R_y = \rho \cdot Q \cdot c \cdot \left(\sin\alpha - \sin\beta\right)$$

After substitution substitution:

Po podstawieniu:

$$c = \frac{4 \cdot Q}{\pi \cdot D^2}$$

We obtain:

Otrzymujemy:

$$R_x = \frac{4 \cdot \rho \cdot Q^2}{\pi \cdot D^2} \cdot (\cos\alpha + \cos\beta)$$

$$R_y = \frac{4 \cdot \rho \cdot Q^2}{\pi \cdot D^2} \cdot (\sin\alpha - \sin\beta)$$

The resultant force is:

Napór wypadkowy wynosi:

$$R = \sqrt{R_x^2 + R_y^2} = \frac{4 \cdot \rho \cdot Q^2}{\pi \cdot D^2} \cdot \sqrt{2 \cdot [1 + \cos(\alpha + \beta)]}$$

The sum of angles is:

Suma kątów wynosi:

$$\alpha + \beta = \frac{\pi}{6} + \frac{\pi}{3} = \frac{\pi}{2}$$

Hence we have:

Wobec czego mamy:

$$R = \frac{4 \cdot \sqrt{2} \cdot \rho \cdot Q^2}{\pi \cdot D^2}$$

After substituting the numerical values we obtain:

Po podstawieniu wartości liczbowych otrzymujemy:

$$R = \frac{4 \cdot \sqrt{2} \cdot 1000 \cdot 0.08^2}{3.1415 \cdot 0.08^2} = 1802[N]$$

Chapter 10 – State of stress in the fluid

Stan naprężenia w płynie

It maybe proved that the tensor of stress in the fluid is symmetrical, i.e.: $\tau_{xy} = \tau_{yx}$

Można udowodnić, że tensor stanu naprężenia w płynie jest tensorem symetrycznym, czyli:

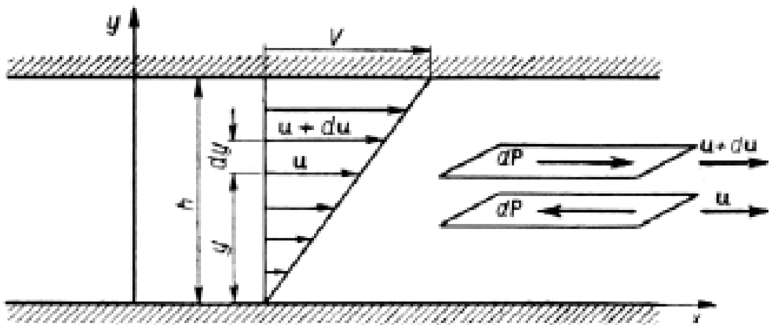
This reduces the number of unknown viscous stresses to 6, which must be determined on the basis of the selected model of the fluid. In most cases the Newtonian model of fluid is employed.

Redukuje to liczbę niewiadomych naprężeń lepkościowych do 6, które muszą być wyznaczone w oparciu o wybrany model płynu. Najczęściej jest stosowany model płynu Newtona.

The Newtonian model of fluid is based on the follow in gas sumptions:
- the fluid is isotropic, i.e. it has the same properties in all directions,
- the stresses in the fluid are linear functions of the rate of strain.

$$\tau_{yx} = \mu \frac{\partial u}{\partial y}$$

Model płynu Newtona oparty jest na następujących założeniach:
- płyn jest izotropowy, czyli ma jednakowe właściwości we wszystkich kierunkach,
- naprężenia w płynie sąliniowymi funkcjami prędkości deformacji

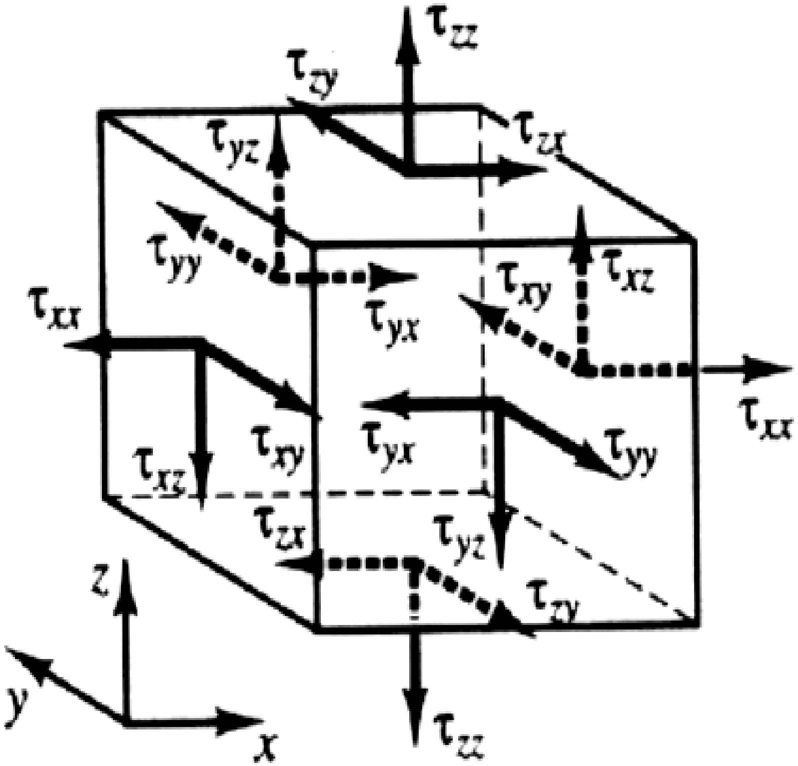


where:

gdzie:

μ the dynamic viscosity coefficient

dynamiczny współczynnik lepkości



The above relations describe the normal and shearing viscous stresses as shown in the picture. As these relations link the stress field to the velocity field, their substitution to the momentum conservation equation reduces the number of unknowns. This is demonstrated in the next lecture.

Powyższe zależności opisująnormalne i styczne naprężenia lepkościowe pokazane na rysunku obok. Ponieważzależności te wiążąpole naprężeńz polem prędkości, ich wstawienie do równania zachowania pędu prowadzi do zredukowania liczby niewiadomych. Jest to pokazane w następnym wykładzie.

In tensor notation we have:

W zapisie tensorowym mamy:

$$[P] = -\left(p + \frac{2}{3}\mu \, div\bar{u}\right)[E] + 2\mu[D]$$

where:

$[E]$ - unit tensor

- tensor jednostkowy

$[D]$ - rate of strain tensor

- tensor prędkości deformacji elementu płynu

In the incompressible fluid we have:

W płynie nieściśliwym mamy:

$$[P] = -p[E] + 2\mu[D]$$

Chapter 11 – Navier – Stokes equation

Substitution of the relations resulting from the Newtonian fluid model into the equation of conservation of the fluid momentum leads to the equation known as the **Navier-Stokes equation.**

This equation may be written in the form of three scalar equations:

$$\rho\frac{Du}{Dt}=\rho f_x-\frac{\partial p}{\partial x}+\frac{\partial}{\partial x}\left[2\mu\frac{\partial u}{\partial x}+\lambda div\bar{u}\right]+\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right]+\frac{\partial}{\partial z}\left[\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\right]$$

$$\rho\frac{Dv}{Dt}=\rho f_y-\frac{\partial p}{\partial y}+\frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right]+\frac{\partial}{\partial y}\left[2\mu\frac{\partial v}{\partial y}+\lambda div\bar{u}\right]+\frac{\partial}{\partial z}\left[\mu\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\right]$$

$$\rho\frac{Dw}{Dt}=\rho f_z-\frac{\partial p}{\partial z}+\frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\right]+\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\right]+\frac{\partial}{\partial z}\left[2\mu\frac{\partial w}{\partial z}+\lambda div\bar{u}\right]$$

In the vector form the Navier - Stokes equation reads:

$$\rho \frac{D\bar{u}}{Dt} = \rho\bar{f} - grad p + grad(\lambda div \bar{u}) + div(2\mu[D])$$

$$A=B+C+D+E$$

A – rate of change of the fluid element momentum

B - mass force

C – surface pressure force

D – surface force connected with fluid viscosity and resulting from the change of volume of the compressible fluid element (compression or expansion)

E – surface force connected with fluid viscosity and resulting from the linear and shearing deformation of the fluid element

In an incompressible fluid the Navier-Stokes equation simplifies to the form:

$$\rho \frac{D\bar{u}}{Dt} = \rho\bar{f} - grad p + div(2\mu[D])$$

If additionally a constant fluid viscosity is assumed, we obtain:

$$\rho \frac{D\bar{u}}{Dt} = \rho\bar{f} - grad p + \mu\Delta\bar{u}$$

Further possible simplification is the assumption of zero viscosity of the fluid, which leads to the **Euler equation**, describing the motion of an incompressible and inviscid fluid:

$$\rho \frac{D\bar{u}}{Dt} = \rho\bar{f} - grad p$$

The Navier-Stokes equation may be solved analytically only for a few simplified cases. Selected examples are described below.

Examples of the analytical solutions of the Navier - Stokes for simple flows

Assumption: we consider an unidirectional flow, i.e. the flow in which *v=w=0,*so the velocity vectors are parallel to each other in every point of the field. If the fluid is incompressible, we get from the mass conservation equation:

$$\frac{\partial u}{\partial x} = 0 \qquad \text{or:} \qquad u = u(y, z, t)$$

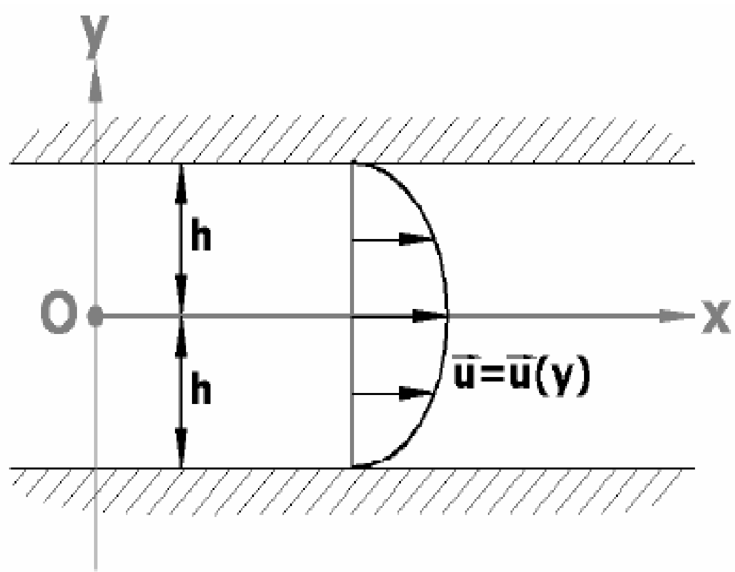
Then the Navier – Stokes equation simplifies to the form:

$$\rho \frac{\partial u}{\partial t} - \mu \left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) = -\frac{\partial p}{\partial x}$$

As the left hand side does not depend on x, and: $\frac{\partial p}{\partial y} = \frac{\partial p}{\partial z} = 0$

then: $p = p(x, t)$ $\quad \frac{dp}{dx} = f(t)$ and finally: $\frac{dp}{dx} = \frac{\Delta p}{\Delta x}$

Example No. 1: Steady laminar flow between two infinite parallel plates (Poiseuille flow) or enforced flow



Given: $\frac{\Delta p}{\Delta x} = const$

Boundary conditions:

$u=0$ for $y=h$

$u=0$ for $y=-h$

Solution

The Navier – Stokes equation takes the form: $$\frac{d^2u}{dt^2} = \frac{1}{\mu}\frac{\Delta p}{\Delta x}$$

After double integration we obtain:

$$u(y) = \frac{1}{\mu}\frac{\Delta p}{\Delta x}\frac{y^2}{2} + C_1 y + C_2$$

The integration constants may be determined from the boundary conditions, what leads to the solution:

$$u(y) = -\frac{1}{2\mu}\frac{\Delta p}{\Delta x}\left(h^2 - y^2\right)$$

The volumetric flow intensity may be determined as follows:

$$Q = \int_S u dS = \int_{-h}^{+h} u(y) dy = -\frac{2}{3\mu}\frac{\Delta p}{\Delta x} h^3$$

Comments:

- the velocity profile is parabolic with maximum at y=0,
- for increasing pressure gradient the maximum velocity and the volumetric flow intensity also increase,
- for increasing fluid viscosity the maximum velocity and the volumetric intensity off low decrease.

Example No.2: Steady laminar flow through the horizontal pipe of constant circular cross-section (case of α=0)

Given: $$\frac{\Delta p}{\Delta x} = const$$

Boundary conditions:

$$u(R) = 0 \qquad u(0) < \infty$$

Solution

We employ the cylindrical system of coordinates, in which the Laplace operator applied to the velocity field has the form:

$$\nabla^2 u = \frac{1}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right)$$

The Navier-Stokes equation is:

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right)=\frac{1}{\mu}\frac{\Delta p}{\Delta x}$$

Double integration with respect to r gives:

$$r\frac{du}{dr}=\frac{1}{\mu}\frac{\Delta p}{\Delta x}\frac{r^2}{2}+C_1 \qquad u=\frac{1}{\mu}\frac{\Delta p}{\Delta x}\frac{r^2}{4}+C_1\ln r+C_2$$

The integration constants may be determined from the boundary conditions:

$$C_1=0 \qquad A=\pi r^2 \; C_2=-\frac{1}{\mu}\frac{\Delta p}{\Delta x}\frac{R^2}{4}$$

Ultimately this leads to:

$$u(r)=\frac{1}{4\mu}\frac{\Delta p}{\Delta x}(R^2-r^2)$$

$$Q=\int_0^R u(r)\cdot 2\pi r dr=-2\pi\cdot\frac{1}{4\mu}\frac{\Delta p}{\Delta x}\int_0^R\left(R^2-r^2\right)dr=-\frac{\pi}{8\mu}\frac{\Delta p}{\Delta x}R^4$$

Comment: comparison with the Example No.1 assuming h=R leads to the conclusion that with the same pressure gradient the maximum flow velocity and the volumetric flow intensity in the pipe are smaller. This is due to the more intensive retardation of the pipe flow by the viscous stresses.

The case of pipe inclined at an angle α to the horizon

In this case the Navier-Stokes equation must include the component of the mass force acting in the direction of flow:

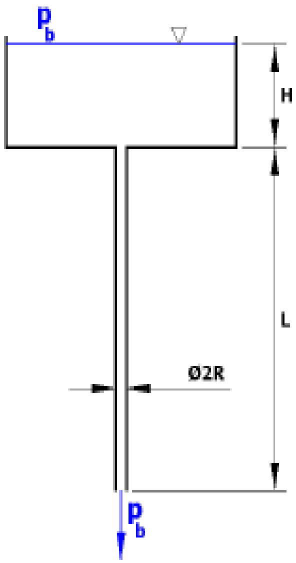
$$\frac{1}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right)=\frac{1}{\mu}\frac{\Delta p}{\Delta x}-\frac{g}{\nu}\sin\alpha$$

As the pressure gradient and the mass flow component act on the flow in a similar way we may introduce the hydraulic head *J*:

$$J=\sin\alpha-\frac{1}{\rho g}\frac{\Delta p}{\Delta x}$$

Then the solution has the form:

$$u(r) = \frac{gJ}{4\nu}\left(R^2 - r^2\right) \qquad Q = \frac{\pi g J}{8\nu} R^4$$



A particular case: the vertically oriented pipe

In this case we have: $$J = 1 - \frac{\Delta p}{\rho g L}$$

Pressure at inlet: $$p_1 = p_b + \rho g H$$

Pressure at outlet: $$p_2 = p_b$$

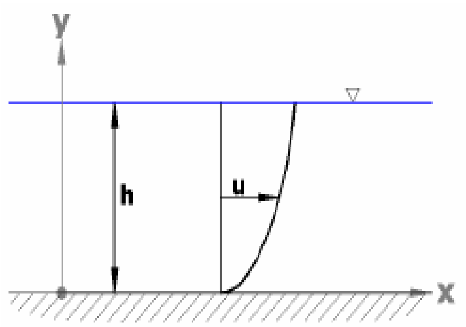
Then: $$\Delta p = p_2 - p_1 = -\rho g H$$

After substitution we get:

$$Q = \frac{\pi g R^4}{8\nu}\left(1 + \frac{H}{L}\right)$$

Due to the easy measurement of Q this formula may be employed for experimental determination of the fluid viscosity coefficient ν

Example No.3: Flow in an open channel



Boundary conditions:

$$\frac{du}{dy}=0 \qquad \text{for: } y=h \qquad (1)$$

$$u=0 \qquad \text{for } y=0 \qquad (2)$$

Solution

$$\frac{d^2u}{dy^2}=\frac{1}{\mu}\frac{\Delta p}{\Delta x} \quad \rightarrow \quad \frac{du}{dy}=\frac{1}{\mu}\frac{\Delta p}{\Delta x}y+C_1$$

From condition (1): $C_1=-\frac{1}{\mu}\frac{\Delta p}{\Delta x}h$ what leads to:

$$\frac{du}{dy}=\frac{1}{\mu}\frac{\Delta p}{\Delta x}y-\frac{1}{\mu}\frac{\Delta p}{\Delta x}h$$

After second integration we get: $$u = \frac{1}{2\mu}\frac{\Delta p}{\Delta x}y^2 - \frac{1}{2\mu}\frac{\Delta p}{\Delta x}hy + C_2$$

$C_2 = 0$ from condition (2), what ultimately leads to:

$$u = \frac{1}{2\mu}\frac{\Delta p}{\Delta x}y(y-2h)$$

And the volumetric intensity of flow is:

$$Q = \int_0^h \frac{1}{2\mu}\frac{\Delta p}{\Delta x}\left(y^2 - 2hy\right)dy = -\frac{2}{3}\frac{\Delta p}{\Delta x}h^3$$

NB!: when comparing the above results with those for the Poiseuille flow the differences in the adopted systems of coordinates should be taken in to account.

Chapter 12 – Energy conservation equation

Kinetic energy of the fluid may be treated as the sum of the macroscopic motion energy and molecular motion energy or internal energy:

$$\int_V \left(\frac{u^2}{2}+e\right)dV \qquad \bar{u}\left(u_x,u_y,u_z\right) \qquad u=\left|\bar{u}\right|$$

Rate of change (i.e. material derivative) of the total kinetic energy of the fluid volume *V* surrounded by the fluid surface *S* is equal to the sum of the power of mass forces, the power of surface forces and the stream of energy (heat) supplied to the element.

$$\frac{D}{Dt}\int_V \rho\left(\frac{u^2}{2}+e\right)dV=\int_V \rho\bar{f}\bullet\bar{u}dV+\int_{S(V)}\bar{\tau}\bullet\bar{u}dS-\int_{S(V)}\bar{j}\bullet\bar{n}dS$$

where:

$\bar{f}$	unit mass force	$\bar{f}\left(f_x,f_y,f_z\right)$
$\bar{\tau}$	unit surface force	
$\bar{j}$	stream of supplied energy (heat)	$\bar{j}\left(j_x,j_y,j_z\right)$
$\bar{n}$	External unit normal vector	

The surface integrals in this equation may be converted into the volume integrals. If simultaneously we express the unit surface forces by the stress tensor **[P]** , the above equation may be converted into the following form:

$$\int_V \left[\rho \frac{D}{Dt}\left(\frac{u^2}{2}+e\right) - \rho\bar{f} \bullet \bar{u} - div\left(\left[P\right]\bar{u}\right) + div\bar{j} \right] dV = 0$$

As the volume of integration is arbitrarily selected, the function under the integral must be also equal to zero, what leads to the integral form of the energy conservation equation:

$$\rho \frac{D}{Dt}\left(\frac{u^2}{2}+e\right) = \rho\bar{f} \bullet \bar{u} + div\left(\left[P\right]\bar{u}\right) - div\bar{j}$$

Full development of the operators in the equation leads to the following form:

$$\rho\frac{D}{Dt}\left(\frac{u^2}{2}+e\right)=\rho\left(f_x u_x+f_y u_y+f_z u_z\right)+\frac{\partial}{\partial x}\left[\left(-p+\tau_{xx}\right)u_x+\tau_{yx}u_y+\tau_{zx}u_z\right]+$$

$$+\frac{\partial}{\partial y}\left[\tau_{xy}u_x+\left(-p+\tau_{yy}\right)u_y+\tau_{zy}u_z\right]+\frac{\partial}{\partial z}\left[\tau_{xz}u_x+\tau_{yz}u_y+\left(-p+\tau_{zz}\right)u_z\right]+$$

$$-\frac{\partial j_x}{\partial x}-\frac{\partial j_y}{\partial y}-\frac{\partial j_z}{\partial z}$$

If the field of mass forces is a stationary potential field such that

$\bar{f}=-grad\Pi$ then the energy conservation equation may be written as:

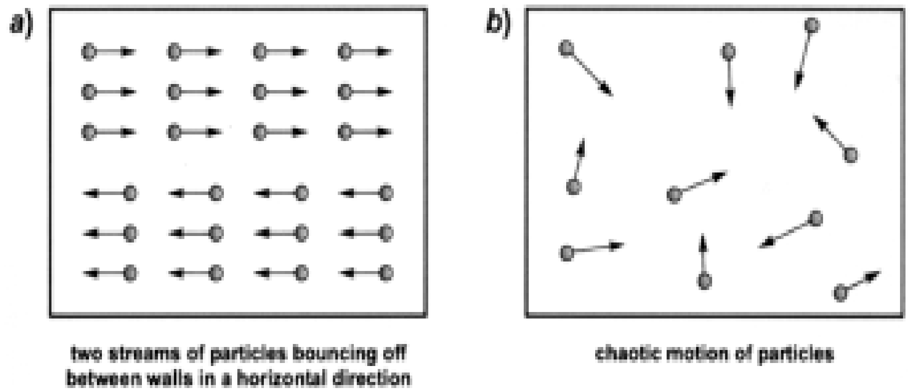
$$\rho\frac{D}{Dt}\left[\left(\frac{u^2}{2}+e+\Pi\right)\right]=div\left([P]\bar{u}\right)-div\bar{j}$$

or in the form of an integral equation:

$$\int_V\frac{\partial}{\partial t}\left[\rho\left(\frac{u^2}{2}+e+\Pi\right)\right]dV+\int_{S(V)}\rho\left(\frac{u^2}{2}+e+\Pi\right)u_n dS=\int_{S(V)}\bar{\tau}\bullet\bar{u}dS-\int_{S(V)}\bar{j}\bullet\bar{n}dS$$

Chapter 13 –Balance of entropy equation

Entropy is the function of the fluid state parameters and it is the measure of chaos in the molecular motion and the measure of „useless” energy of the given system.



a) low entropy system b) high entropy system

Unit of entropy S $\left[\frac{J}{K}\right]$

Unit of the specific entropy s $\left[\frac{J}{kg \cdot K}\right]$

Features of entropy

Entropy is transported with heat according to the Clausius formula:

$$j_s = \frac{1}{T} j$$

where:

- j_s stream of entropy
- j stream of heat
- T temperature at which transport takes place

Entropy changes with the fluid state parameters (Gibbs formula):

$$T\frac{Ds}{Dt} = \frac{De}{Dt} + p\frac{D}{Dt}\left(\frac{1}{\rho}\right)$$

where:

- p pressure
- e fluid internal energy
- ρ fluid density

The second law of thermodynamics: in any real process the sum of changes of entropy of all the bodies taking part in the process is always positive.

The rate of change (i.e. the material derivative) of entropy in the fluid volume *V(S)* is equal to the production of entropy in side this volume and the stream of entropy through the fluid surface *S.*

Where: $\dot{s}$ volumetric intensity of the entropy sources

$$\frac{D}{Dt}\int_V \rho s dV = \int_V \dot{s} dV - \int_{S(V)} \bar{j}_s \bullet \bar{n} dS$$

The above equation may be converted into the form of a single volumetric integral:

$$\int_V \left(\rho \frac{Ds}{Dt} - \dot{s} + div \frac{\bar{j}}{T} \right) dV = 0$$

As the fluid volume V was arbitrarily selected, the function under the intergal must be also equal zero, what leads to the balance of entropy equation in the differential form (i.e. for a fluid element):

$$\rho \frac{Ds}{Dt} = \dot{s} - div \frac{\bar{j}}{T}$$

If we use the Gibbs formula, we obtain:

$$\dot{s} = \frac{\rho}{T}\frac{De}{Dt} - \frac{p}{\rho T}\frac{D\rho}{Dt} + div\frac{\bar{j}}{T}$$

The above equation may be transformed using the equations of mass conservation, momentum conservation, energy conservation and the thermal conductivity theorem of Fourier:

$$\dot{s} = \dot{s}_M + \dot{s}_T = \frac{\mu}{T}\left[\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)^2 + \left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right)^2 + \left(\frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z}\right)^2 + \right.$$
$$\left. + \frac{2}{3}\left(\frac{\partial u_x}{\partial x} - \frac{\partial u_y}{\partial y}\right)^2 + \frac{2}{3}\left(\frac{\partial u_x}{\partial x} - \frac{\partial u_z}{\partial z}\right)^2 + \frac{2}{3}\left(\frac{\partial u_y}{\partial y} - \frac{\partial u_z}{\partial z}\right)^2\right] + \frac{\lambda}{T^2}(gradT)^2$$

Fourier theorem: $\bar{j} = -\lambda gradT$

The balance of entropy equation in the above form illustrates the continuous process of **dissipation of the mechanical energy** of the flowing fluid and its conversion into heat.

Remarks:

- both terms of the intensity of the volumetric sources of entropy are always non-negative,
- mechanical sources of entropy are equal zero when μ=0, and thermal sources of entropy are equal zero when λ=0 , what leads to the model of inviscid and non-conducting fluid,
- the above equation shows that the internal fluid energy depends on:

a) entropy processes (combustion, chemical reactions, internal friction of the fluid),
b) change of fluid density (compression or expansion),
c) addition or subtraction of heat.

Chapter 14 – The closed system of equations of the fluid mechanics

Zamknięty układ równań mechaniki płynów

The above presented equations form the closed system of the fluid mechanics equations, which may be employed for description of realistic flows and for obtaining, through solution of these equations, information about the values of the interesting parameters describing these flows. The actual format of the system of equations depends on the adopted fluid model and flow model.

Przedstawione powyżej równania tworzą zamknięty układ równań mechaniki płynów, który może być zastosowany do opisu konkretnych przepływów i uzyskania, w drodze rozwiązania tego układu, informacji o wartościach interesujących nas parametrów tego przepływu. Konkretna postać układu równań zależy od przyjętego modelu płynu.

Case No. 1: Incompressible fluid of constant viscosity
The closed system of equations is formed of:

Przypadek 1: płyn nieściśliwy o stałej lepkości
Zamknięty układ równań tworzą:

- mass conservation equation
- równanie zachowania masy

$$div\bar{u} = 0$$

-momentum conservation equation
- równanie zachowania pędu

$$\rho \frac{D\bar{u}}{Dt} = \rho\bar{f} - grad p + \mu \Delta \bar{u}$$

These are equivalent to four scalar equations with four unknowns:
Razem są to cztery równania skalarne z czterema niewiadomymi:

- pressure p
- ciśnienie
- velocity components u_x, u_y, u_z
- składowe prędkości

In this case the temperature field does not influence the flow, but it depends it self on the velocity field of the flow through the entropy balance equation in the form:

W tym przypadku pole temperatury nie wpływa na przepływ, ale samo jest uzależnione od pola prędkości przepływu poprzez równanie bilansu entropii w postaci:

$$\rho c\left(\frac{\partial T}{\partial t}+u_x\frac{\partial T}{\partial x}+u_y\frac{\partial T}{\partial y}+u_z\frac{\partial T}{\partial z}\right)=Ts_M+\lambda\Delta T$$

This form of the equation may be obtained from the original formula by substituting the relation for the fluid internal energy:

Tę postać równania można uzyskać podstawiając do oryginalnego równania zależność dla energii wewnętrznej:

$$e=cT+e_0$$

In the case when the fluid viscosity depends on the temperature, the balance of entropy equation is connected with the mass and momentum conservation equations through the relation:

W przypadku gdy lepkość płynu zależy od temperatury, równanie bilansu entropii jest sprzężone z równaniami zachowania masy i zachowania pędu poprzez zależność:

$$\mu=\mu(T)$$

Then we have the system of six equations with six unknowns:

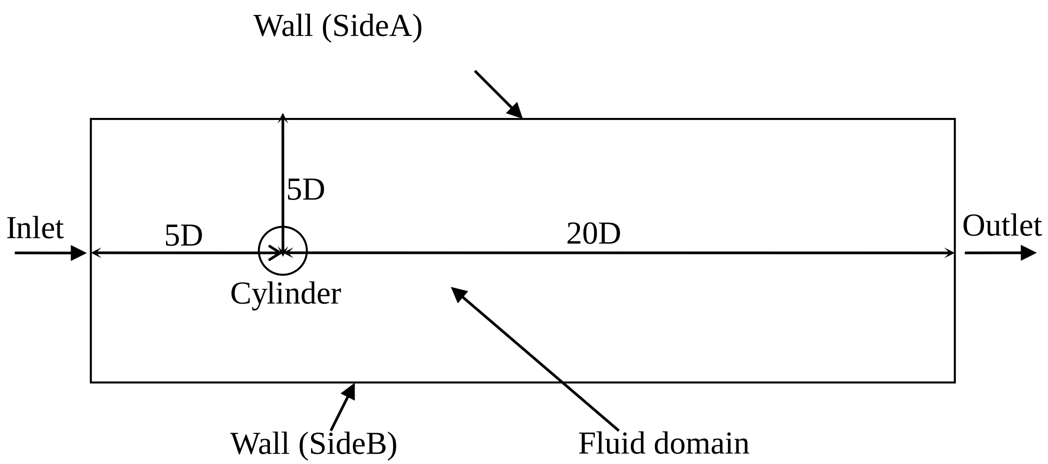
Mamy wtedy układ sześciu równań z sześcioma niewiadomymi:

- pressure p
  - ciśnienie
- temperature T
  - temperatura
- velocity components u_x, u_y, u_z
  - składowe prędkości
- viscosity coefficient μ
  - współczynnik lepkości

Flow around a circular cylinder

Problem Description

Air flows across a cylinder with the uniform velocity *0.003m/s* in the wind tunnel. The length of the wind tunnel (fluid domain) has *25m* long and *10 m* height. The diameter of cylinder is *1m*.



Assumption and Boundary Conditions:

1. Dimensional problems
2.Steady state condition
3.The uniform flow velocity
4.No Heat transfer
5.Neglect the gravitational force
6.Constant air density

Case No. 2: compressible fluid

Przypadek 2: płyn ściśliwy

In this case the closed system of equations is formed of:

W tym przypadku zamknięty układ równań tworzą:

- mass conservation equation
  -równanie zachowania masy

$$\frac{\partial \rho}{\partial t} + div(\rho \overline{u}) = 0$$

- momentum conservation eq.
  - równanie zachowania pędu

$$\rho \frac{D\overline{u}}{Dt} = \rho \overline{f} - grad p - grad\left(\frac{2}{3}\mu div \overline{u}\right) + div(2\mu[D])$$

- entropy balance equation
  - równanie bilansu entropii

$$\rho \frac{De}{Dt} = T\dot{s}_M + \frac{p}{\rho}\frac{Dp}{Dt} + \lambda \Delta T$$

- internal energy equation
  - równanie energii wewnętrznej

$$e = \int_{T_0}^{T} c_v(T) dT$$

- equation of state
  - równanie stanu

$$\frac{p}{\rho} = Z(p,T)RT$$

- additional relations
  - dodatkowe zależności

Z – compressibility function
– funkcja ściśliwości

R – gas constant
– stała gazowa

$$\mu = \mu(T) \qquad c_V = c_V(T)$$

In this case we have the system of nine equations with nine unknowns:
W tym przypadku mamy układ dziewięciu równań z dziewięcioma niewiadomymi:

- pressure p / - ciśnienie
- density ρ / - gęstość
- temperature T / - temperatura
- internal energy e / - energia wewnętrzna
- viscosity coefficient μ / - współczynnik lepkości
- velocity components u_x, u_y, u_z / - składowe prędkości
- specific heat c_V / - ciepło właściwe

It was assumed that the thermal conductivity coefficient λ is constant and given.
Założono, że współczynnik przewodnictwa cieplnego λ ma wartość stałą.

Boundary and initial conditions

In order to enable solution of the above systems of equations it is necessary to determine the appropriate boundary and (for unsteady flows) initial conditions. These conditions are required to determine the arbitrary constants and arbitrary functions resulting from the integration of the equations.

Warunki brzegowe i początkowe

Dla umożliwienia rozwiązania powyższych układów równań konieczne jest określenie odpowiednich warunków brzegowych oraz (dla przepływów niestacjonarnych) warunków początkowych. Warunki te są potrzebne do wyznaczenia dowolnych stałych i dowolnych funkcji wprowadzonych podczas całkowania równań.

Boundary conditions

Boundary conditions on the surface of an impermeable body

Warunki brzegowe

Warunki brzegowe na powierzchni ciała stałego nieprzzenikliwego

- inviscid fluid –normal velocity equal zero
- płyn nielepki –prędkość normalna równa zero

$$u_n = 0$$

- viscous fluid – total velocity equal zero
- płyn lepki –prędkość równa zero

$$\overline{u} = 0$$

-given temperature *T* or stream of heat *j*
- dana temperatura *T* lub strumień ciepła *j*

Boundary conditions on the surface of a porous body

Warunki brzegowe na powierzchni ciała stałego porowatego

- tangential velocity equal zero
- prędkość styczna równa zero

$$u_t = 0$$

- normal velocity given
- prędkość normalna zadana

$$u_n = f(x, y, z)$$

- given temperature *T* or stream of heat *j*
- dana temperatura *T* lub strumień ciepła *j*

Boundary conditions on the free surface between two fluids

Warunki brzegowe na powierzchni rozdziału dwóch płynów

- inviscid fluid
- płyn nielepki

$$u_n^1 = u_n^2$$

- viscous fluid
- płyn lepki

$$\overline{u}_1 = \overline{u}_2$$

If the free surface equation between two fluids is described by:

Jeżeli równanie powierzchni rozdziału płynów ma postać:

$$F(x, y, z, t) = 0$$

Then the kinematic boundary condition has the form:

To kinematyczny warunek brzegowy ma postać:

$$\frac{\partial F}{\partial t} + u_x \frac{\partial F}{\partial x} + u_y \frac{\partial F}{\partial y} + u_z \frac{\partial F}{\partial z} = 0$$

Boundary conditions a infinity are given in the case of flow around an object in which the velocity field far from the object is uniform.

Warunki w nieskończonościzadaje się w przypadku opływu obiektu strugą, w której pole prędkości z dala od obiektu jest jednorodne.

$$\bar{u} = \bar{u}(\infty) \qquad p = p(\infty) \qquad T = T(\infty)$$

Initial conditions

The initial conditions refer to the unsteady flow phenomena and they should be determined for every point of space filled with the fluid for the initial instant of time $t = t_0$

Warunki początkowe

Warunki początkowe dotyczą zjawisk niestacjonarnych i powinny być podane dla wszystkich punktów przestrzeni wypełnionej płynemdla chwili $t = t_0$

For the incompressible fluid flow:
Dla przepływu płynu nieściśliwego:

$$p = p(x, y, z, t_0) \qquad \bar{u} = \bar{u}(x, y, z, t_0)$$

For the incompressible fluid flow with viscosity dependent on temperature, additionally:

Dla przepływu płynu nieściśliwego o lepkości zależnej od temperatury dodatkowo:

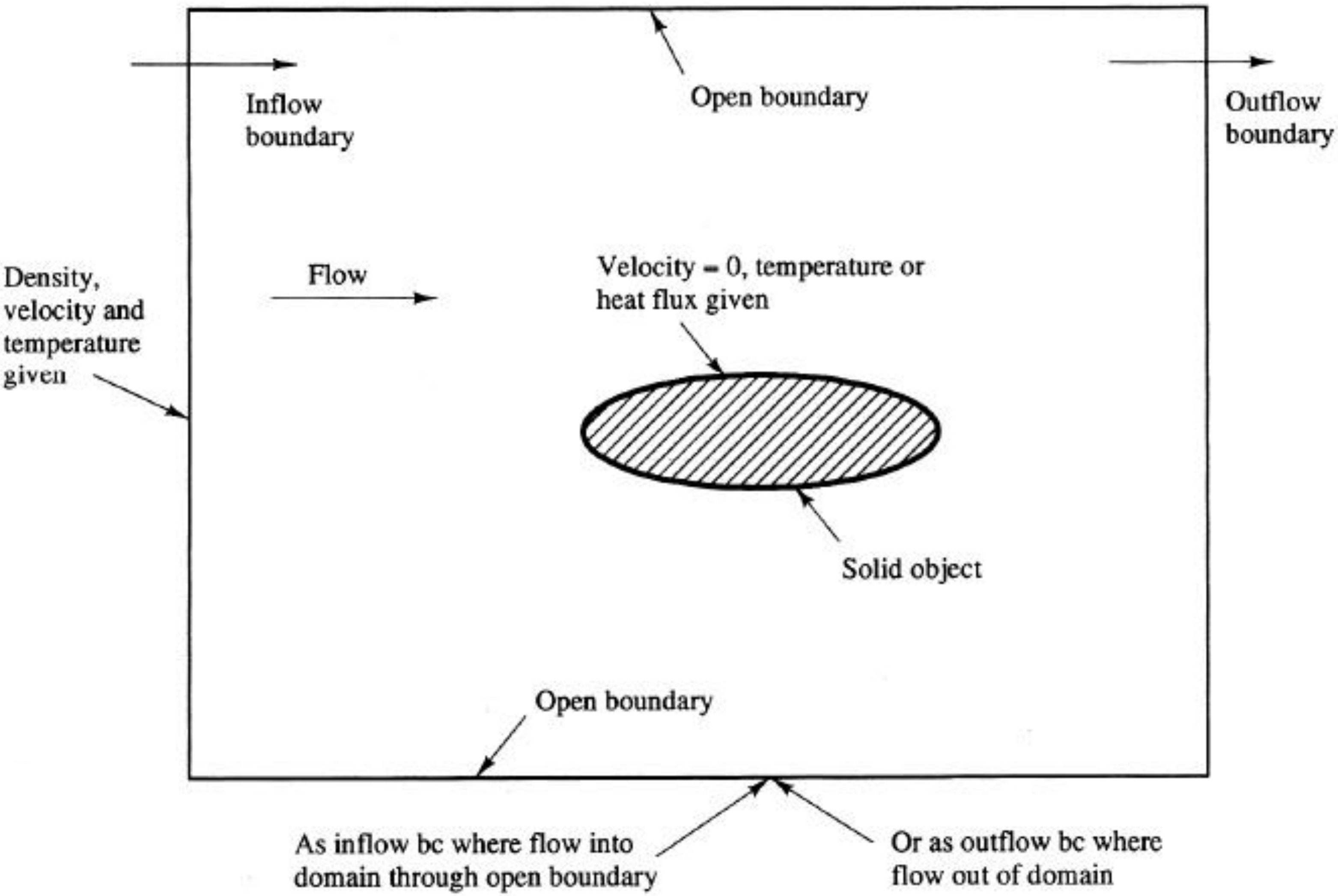
$$T = T(x, y, z, t_0) \qquad \mu = \mu(x, y, z, t_0)$$

For the compressible fluid flow, additionally:

Dla przepływu płynu ściśliwego dodatkowo:

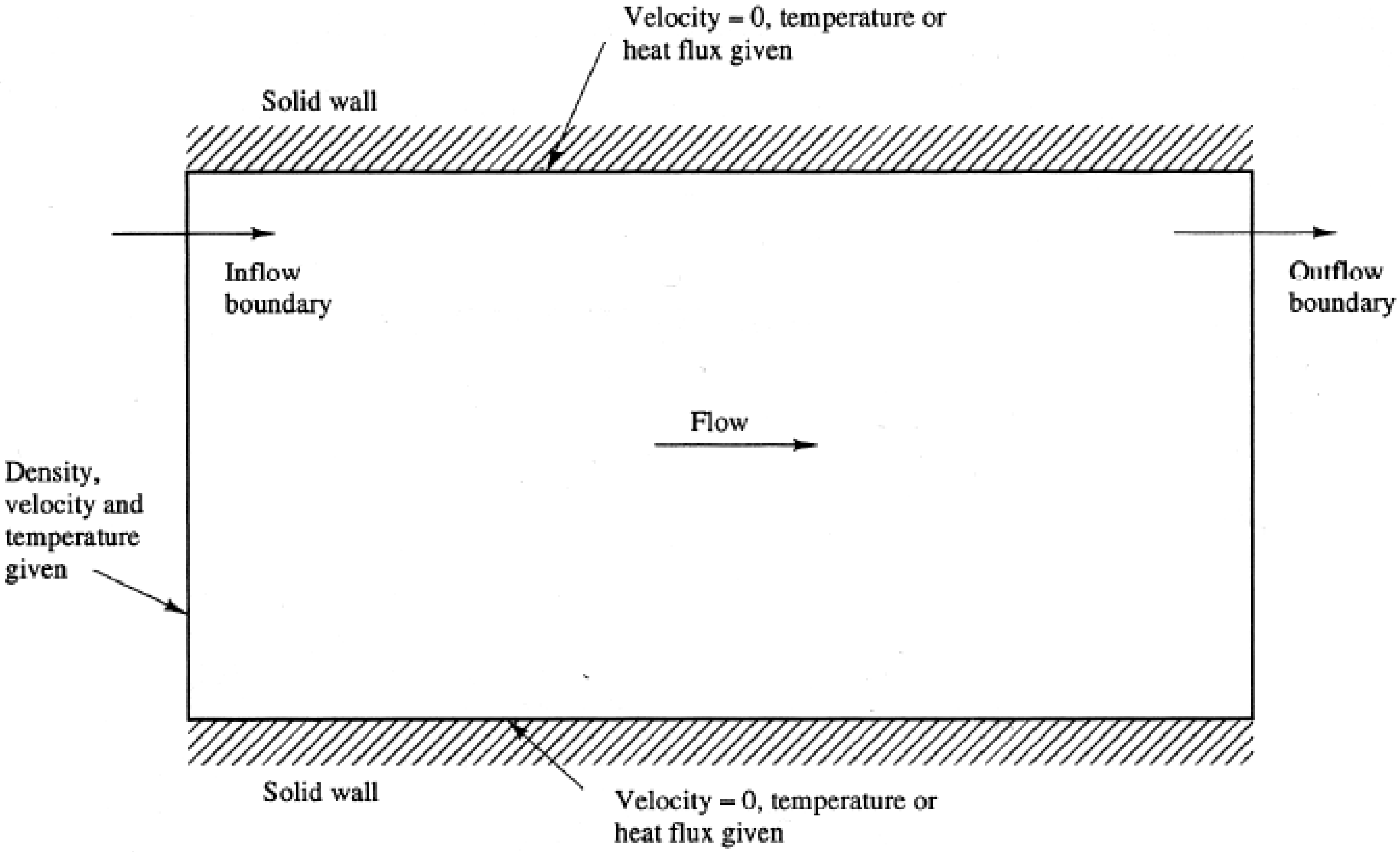
$$\rho = \rho(x, y, z, t_0) \qquad e = e(x, y, z, t_0) \qquad c_V = c_V(x, y, z, t_0)$$

The initial conditions should not be in conflict with the boundary conditions.
Warunki początkowe powinny być niesprzeczne z warunkami brzegowymi.



Scheme of the boundary conditions for an external flow

Schemat warunków brzegowych dla przepływu zewnętrznego



Scheme of the boundary conditions for an internal flow

Schemat warunków brzegowych dla przepływu zewnętrznego

Chapter 15 – Bernoulli equation

Równanie Bernoulliego

Bernoulli equation expresses, under certain assumptions, the principles of momentum conservation and energy conservation of the fluid.

Równanie Bernoulliego wyraża zasady zachowania pędu i zachowania energii płynu przy spełnieniu odpowiednich założeń.

Assumptions:

Założenia:

- the flow is stationary $\frac{\partial}{\partial t} = 0$

- przepływ jest stacjonarny

- the fluid is inviscid $\mu = 0$

-plyn jest nielepki

- the fluid is barotropic $\rho = \rho(p)$

- płyn jest barotropowy

- the mass forces form a potential field $\bar{f} = -grad\Pi$

- pole sił masowych jest potencjalne

Under such assumptions the Euler equation may be integrated:

Przy takich założeniach można scałkować równanie Eulera:

$$\rho \frac{D\bar{u}}{Dt} = \rho\bar{f} - gradp$$

We use the identity:

Korzystamy z tożsamości:

$$\frac{D\bar{u}}{Dt} = \frac{\partial \bar{u}}{\partial t} + \bar{u}\, grad\, \bar{u} = \frac{\partial \bar{u}}{\partial t} + grad \frac{u^2}{2} + rot\bar{u} \times \bar{u}$$

Moreover, we introduce a pressure function:

Ponadto wprowadzamy funkcję ciśnienia:

$$P(p) = \int_{p_0}^{p} \frac{dp}{\rho(p)}$$

what leads to:

Co prowadzi do:

$$\frac{1}{\rho} grad p = grad P(p)$$

Then the Euler equation may be written as:

Wtedy równanie Eulera można napisać w postaci

$$\frac{\partial \bar{u}}{\partial t} + grad\left[\frac{u^2}{2} + P(p) + \Pi\right] = rot\bar{u} \times \bar{u}$$

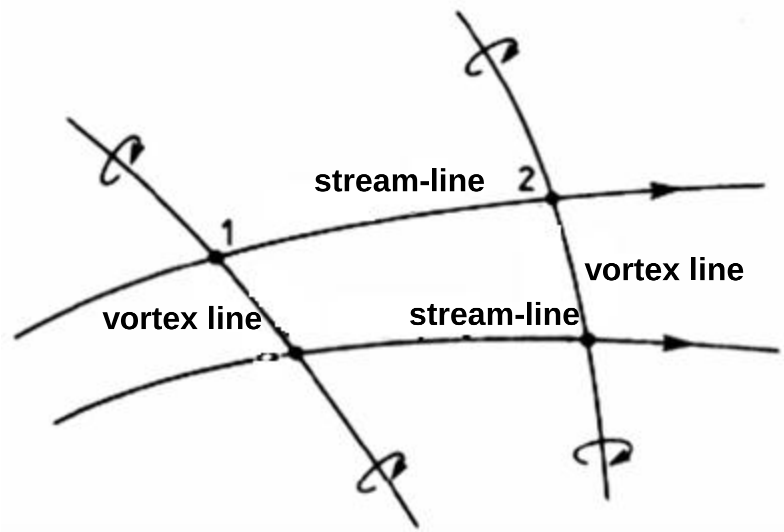
or:

Czyli:

$$grad\left[\frac{u^2}{2} + P(p) + \Pi\right] = grad E = rot\bar{u} \times \bar{u}$$

The expression in brackets is known as the Bernoulli tri-nomial *E*. It may be proved that *E* is constant in five cases.

Wyrażenie w nawiasie nazywamy trójmianem Bernoulliego *E*. Można wykazać stałość tego trójmianu w pięciu przypadkach.



Case1: a long a stream-line

Przypadek 1: wzdłuż linii prądu

$$gradE = \bar{u} \times rot\bar{u}$$

Both sides are multipled by the unit length vector a long the steam-lines $\bar{i}_s$

Mnożymy obie strony skalarnie przez wersor linii prądu

This leads to: $\dfrac{dE}{ds} = 0$ or: $E = const$

Co prowadzi do: czyli:

Case 2: a long a vortex line

Both sides are multiplied by the unit length vector along the vortex line

Przypadek 2: wzdłuż linii wirowej

Mnożymy obie strony równania skalarnie przez wersor linii wirowej

$$gradE \bullet \bar{i}_\omega = \left(\bar{u} \times rot\bar{u}\right) \bullet \bar{i}_\omega = 0$$

this gives (co daje) $\dfrac{dE}{d\omega} = 0$ or (czyli) $E = const$

Stream-lines and vortex lines form so called Bernoulli surface, at which there is

Linie prądu i linie wirowe tworzą tzw. powierzchnię Bernoulliego na której jest $E = const.$

Case 3: an irrotational flow

Przypadek 3: w przepływie bezwirowym

$$rot\bar{u} = 0 \rightarrow E = const$$

Case 4: a hydrostatic situation

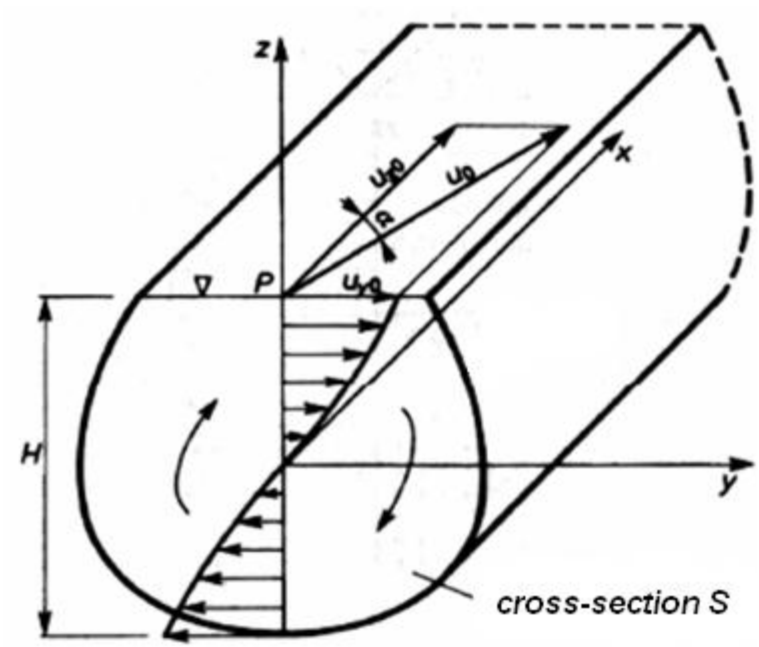
Przypadek 4: w sytuacji hydrostatycznej

$$\bar{u} = 0 \rightarrow E = const$$

Case 5: a helicoidal flow

Przypadek 5: w przepływie śrubowym

$$rot\bar{u} = \lambda\bar{u}$$



$$\bar{u} \times rot\bar{u} = \bar{u} \times \lambda\bar{u} = 0 \rightarrow E = const$$

In the case of an **incompressible fluid** flow in the **gravitational field** we have:

W przypadku przepływu płynu **nieściśliwego**w **polu grawitacyjnym mamy:**

$$\rho = const$$

and
oraz

$$\Pi = gz$$

what gives:
co daje:

Other forms of the Bernoulli equation are possible if particular forms of the barotropic relation are adopted. For example, in case of a gas under going an adiabatic process this relation reads:

Możliwe są inne postaci równania Bernoulliego, jeżeli przyjmie się szczególne formy warunku barotropowościpłynu. Na przykład dla gazu podlegającego przemianie adiabatycznej warunek ten ma postać:

$$\rho = \frac{\rho_0}{p_0^{1/\kappa}} p^{1/\kappa}$$

where κ jest is the Poisson adiabatic exponent $\kappa = \frac{c_p}{c_v}$

gdzie *κ* jest wykładnikiem adiabaty Poissona

Then the Bernoulli equation takes the form:

Równanie Bernoulliego przyjmuje wtedy postać:

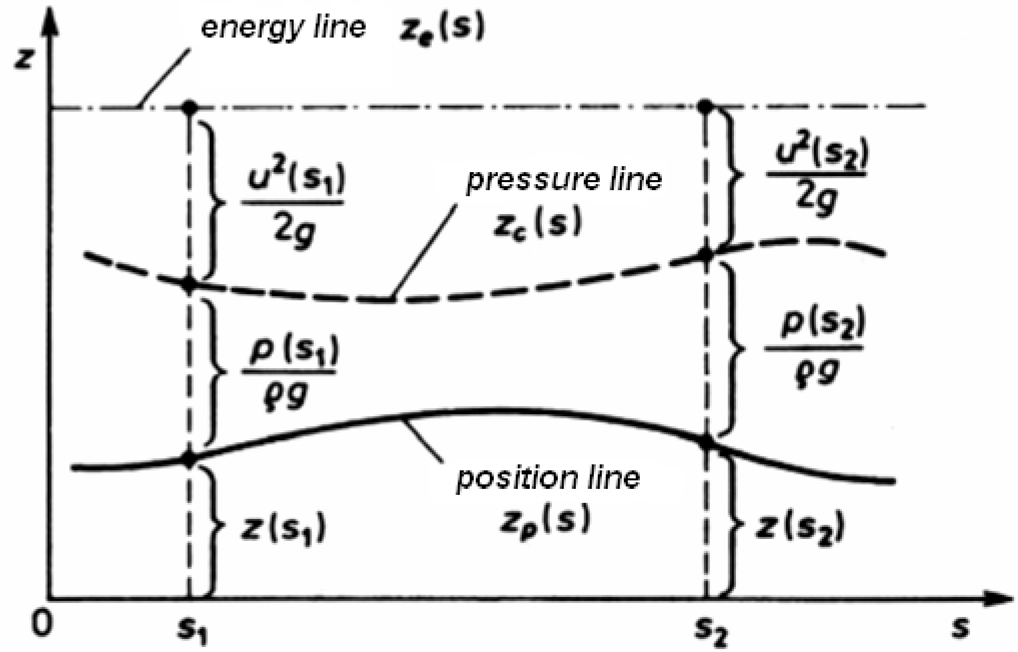
$$\frac{u^2}{2} + \frac{\kappa}{\kappa - 1}\frac{p_0}{\rho_0}\left[\left(\frac{p}{p_0}\right)^{(\kappa-1)/\kappa} - 1\right] + gz = const$$

Comparison of the Bernoulli equation development with the energy conservation equation for a stream tube shows that, with disregarding the fluid internal energy *e* and the thermal conductivity of the fluid, the Bernoulli equation describes the energy conservation principle as well.

Porównanie wyprowadzenia równania Bernoulliego z równaniem zachowania energii dla rurki prądu pozwala stwierdzić, że przy pominięciu energii wewnętrznej płynu *e*i przewodnictwa cieplnego płynu równanie Bernoulliego wyraża również zasadę zachowania energii.

The Bernoulli Equation (1738)

Równanie Bernoulliego (1738)



$$gz + \frac{p}{\rho} + \frac{u^2}{2} = const$$

$$z + \frac{p}{\rho g} + \frac{u^2}{2g} = const$$

$$gz + \frac{p}{\rho} + \frac{u^2}{2} = const$$

The sum of the potential energy of the mass forces, the pressure energy and the kinetic energy of the fluid is constant.

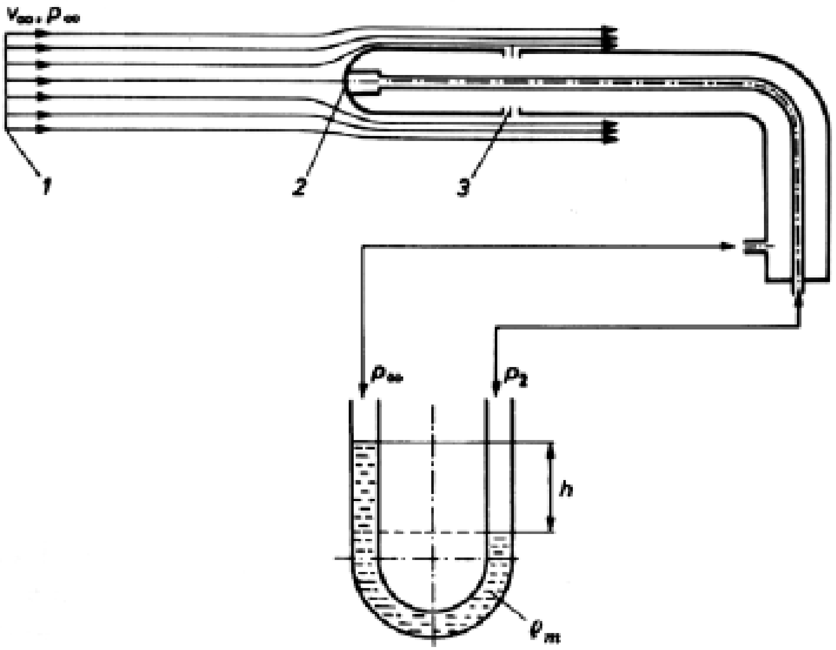
Suma energii potencjalnej pola sił masowych, energii ciśnienia oraz energii kinetycznej płynu jest stała.

or: lub:

The sum of the geometrical elevation, the pressure head (i.e. the height at the fluid is elevated under pressure *p*) and the velocity head (i.e. the height from which the falling fluid achieves velocity *u*) is constant.

Suma wysokości geometrycznej, wysokości ciśnienia (czyli wysokości, na jaką wzniesie się słup cieczy pod ciśnieniem *p*) oraz wysokości prędkości (czyli wysokości, z której spadający elementpłynu uzyska prędkość *u*) jest stała.

Example No. 1



What is the velocity of a fluid flow measured by the Prandtl tube, if the connected to it manometer shows the level difference h of the manometric fluid of density?

The Bernoulli equation for the points 2 and 3 has the form

$$\frac{u_2^2}{2}+\frac{p_2}{\rho}=\frac{u_3^2}{2}+\frac{p_3}{\rho}$$

where: $u_2=0$ $\quad u_3=v_\infty$ $\quad p_3=p_\infty$

What leads to: $$\frac{v_\infty^2}{2}=\frac{p_2-p_\infty}{\rho}$$

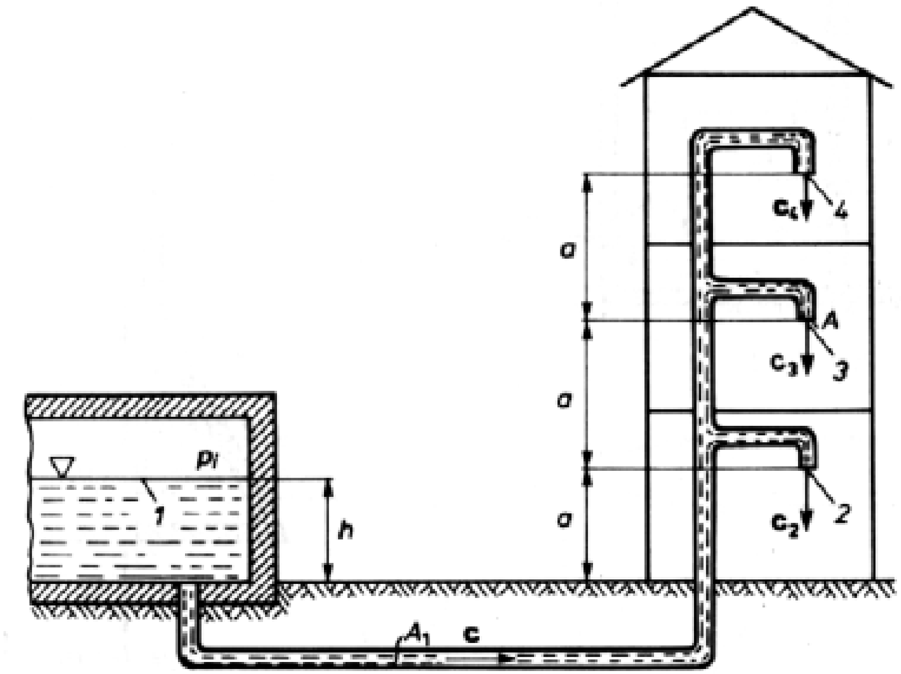
In turn the pressure difference on the manometer is:

$$p_2-p_\infty=\rho_m gh$$

or: $$\frac{v_\infty^2}{2}=\frac{\rho_m}{\rho}gh$$

and finally: $$v_\infty=\sqrt{2\frac{\rho_m}{\rho}gh}$$

Example No. 2



A building is supplied with water from a large tank in which the absolute pressure p_i is and the distance of the water level from the foundation is *h=const.* Determine the velocity of flow on each floor and the velocity of flow in the under ground pipe of cross-section A_1

Given: all cross-sections of pipes in the building are equal *A,* the density of water is equal *ρ,* and the barometric pressure is equal p_b

The Bernoulli equations with respect to the foundation level are :

$$\frac{c_1^2}{2}+\frac{p_i}{\rho}+gh=\frac{c_2^2}{2}+\frac{p_b}{\rho}+ga$$

$$\frac{c_1^2}{2}+\frac{p_i}{\rho}=\frac{c_3^2}{2}+\frac{p_b}{\rho}+2ga$$

$$\frac{c_1^2}{2}+\frac{p_i}{\rho}=\frac{c_3^2}{2}+\frac{p_b}{\rho}+3ga$$

The velocity of the water level in the tank is equal zero, hence we have:

$$c_1 = \sqrt{2\left[\frac{p_i - p_b}{\rho} + g(h-a)\right]}$$

$$c_2 = \sqrt{2\left[\frac{p_i - p_b}{\rho} + g(h-2a)\right]}$$

$$c_3 = \sqrt{2\left[\frac{p_i - p_b}{\rho} + g(h-3a)\right]}$$

The velocity *c* in the under ground tube may be determined from the continuity condition:

what gives:

$$cA_1 = c_2 A + c_3 A + c_4 A$$

$$c = \frac{A}{A_1}(c_2 + c_3 + c_4)$$

Chapter 16 – Similarity of flows I

Experimental testing of flows is most frequently performed on models, which are reduced copies of real objects, manufacture data given scale. In order to produce valid experimental results for the real object, certain similarity conditions between the model and real flows must be observed.

System of measuring of the physical quantities

Basic units	Derived units	
Length [m]	Force	$[N] = \left[kg \frac{m}{s^2} \right]$
Mass [kg]		
Time [s]	Power	$[W] = \left[kg \frac{m^2}{s^3} \right]$
Temperature [K]		

The Buckingham theorem (Π theorem)

1. Every function of *n* dimensional parameters, of which *k* have basic units of measure, may be expressed as the function of *n*-*k* non-dimensional parameters of the form: Formulation of the laws of physics does not depend on the units.

$$\Pi_{k+1} = \frac{a_{k+1}}{a_1^{p_1} a_2^{p_2} \ldots\ldots a_k^{p_k}}$$

2. If the non-dimensional parameters Π are identical for two different situations (e.g. in two different scales), the phenomena are identical in both situations, despite the differences in the dimensional parameters *a*. Consequently, the parameters Π may be regarded as **similarity parameters** or **criteria of similarity**.

On the basis of the above theorem a dimensional analysis of the fluid mechanics equations may be performed and the corresponding criteria of similarity may be developed.

Dimensional analysis of the mass conservation equation

$$\frac{\partial \rho}{\partial t}+\frac{\partial(\rho u_x)}{\partial x}+\frac{\partial(\rho u_y)}{\partial y}+\frac{\partial(\rho u_z)}{\partial z}=0$$ for scale 1

$$\frac{\partial \rho'}{\partial t'}+\frac{\partial(\rho' u'_x)}{\partial x'}+\frac{\partial(\rho' u'_y)}{\partial y'}+\frac{\partial(\rho' u'_z)}{\partial z'}=0$$ for scale 2

We introduce scale coefficients: $\rho' = \alpha_\rho \rho$ $\quad t' = \alpha_t t$

$x' = \alpha_x x \quad y' = \alpha_y y \quad z' = \alpha_z z \quad u'_x = \alpha_{ux} u_x \quad u'_y = \alpha_{uy} u_y \quad u'_z = \alpha_{uz} u_z$

We assume geometrical similarity of flows in both scales, that is:

$$\alpha_x = \alpha_y = \alpha_z = \alpha_l = \frac{l'}{l}$$

Moreover, we assume kinematical similarity, i.e. the similarity of the velocity fields in flows in both scales, that is:

$$\alpha_{ux} = \alpha_{uy} = \alpha_{uz} = \alpha_u = \frac{\overline{u}'}{\overline{u}}$$

Now the equation in scale 2 may be written in the form:

$$\frac{\alpha_\rho}{\alpha_t}\frac{\partial\rho}{\partial t}+\frac{\alpha_\rho\alpha_u}{\alpha_l}\left[\frac{\partial(\rho u_x)}{\partial x}+\frac{\partial(\rho u_y)}{\partial y}+\frac{\partial(\rho u_z)}{\partial z}\right]=0$$

The condition for identity of the equation in scales1 and 2 is:

$$\frac{\alpha_\rho}{\alpha_t}=\frac{\alpha_\rho\alpha_u}{\alpha_l} \quad \text{or} \quad \frac{\alpha_l}{\alpha_t\alpha_u}=1$$

Hence we have the equation: $$\frac{l}{tu}=\frac{l'}{t'u'}=\frac{t_c}{t}=Sh$$

Sh – Strouhal number

t_c - the characteristic time of flow (i.e. the time, in which the fluid covers the characteristic distance *l* – e.g. pipe length, with the characteristic velocity *u*)

t - the time of variation of the unsteady flow parameters, e.g. the time cycle of the piston pump operation

Using the Strouhal number, the mass conservation equation may be written in a non-dimensional form:

$$Sh\frac{\partial\hat{\rho}}{\partial t}+\frac{\partial\left(\hat{\rho}\hat{u}_x\right)}{\partial\hat{x}}+\frac{\partial\left(\hat{\rho}\hat{u}_y\right)}{\partial\hat{y}}+\frac{\partial\left(\hat{\rho}\hat{u}_z\right)}{\partial\hat{z}}=0$$

where all quantities are related to the corresponding characteristic quantities, what makes them non-dimensional, for example:

$$\hat{\rho}=\frac{\rho}{\rho_0} \qquad \hat{t}=\frac{t}{t_0} \qquad \hat{u}_x=\frac{u_x}{u_0} \qquad \hat{x}=\frac{x}{x_0}$$

A small value of the Strouhal number in a given flow means that the non-stationary phenomena in this flow are meaningless and they may be neglected.

Dimensional analysis of the Navier–Stokes equation

Additionally, the following scale coefficient must be introduced:

$$\bar{f}' = \alpha_f \bar{f} \qquad \mu' = \alpha_\mu \mu \qquad p' = \alpha_p p$$

After substitution to the N-S equation we obtain:

$$\frac{\alpha_\rho \alpha_u}{\alpha_t} \rho \frac{\partial \bar{u}}{\partial t} + \frac{\alpha_\rho \alpha_u^2}{\alpha_l} \rho \left[u_x \frac{\partial \bar{u}}{\partial x} + u_y \frac{\partial \bar{u}}{\partial y} + u_z \frac{\partial \bar{u}}{\partial z} \right] = \alpha_\rho \alpha_f \rho \bar{f} - \frac{\alpha_p}{\alpha_l} grad p +$$

$$- \frac{\alpha_\mu \alpha_u}{\alpha_l^2} grad \left(\frac{2}{3} \mu div \bar{u} \right) + \frac{\alpha_\mu \alpha_u}{\alpha_l^2} div (2\mu [D])$$

This equation is identical in two different scales1 and 2 under the following condition:

$$\frac{\alpha_\rho \alpha_u}{\alpha_t} = \frac{\alpha_\rho \alpha_u^2}{\alpha_l} = \alpha_\rho \alpha_f = \frac{\alpha_p}{\alpha_l} = \frac{\alpha_\mu \alpha_u}{\alpha_l^2}$$

After dividing both sides by the second term and using the scale coefficients we get:

Strouhal number:

$$Sh = \frac{l}{tu} = \frac{l'}{t'u'}$$

Froude number:

$$(Fr)^2 = \frac{u^2}{fl} = \frac{u'^2}{f'l'}$$

Froude number defines the ratio of inertia forces and mass forces

Euler number:

$$Eu = \frac{p}{\rho u^2} = \frac{p'}{\rho' u'^2}$$

Euler number defines the ratio of pressure forces to inertia forces

Reynolds number:

$$\mathrm{Re} = \frac{\rho u l}{\mu} = \frac{\rho' u' l'}{\mu'}$$

Reynolds number defines the ratio of inertia forces to viscosity forces

Using the Strouhal, Froude, Eulera and Reynolds numbers the Navier–Stokes equation may be written in a non-dimensional form:

$$Sh\rho\frac{\partial\overline{u}}{\partial t}+\rho(\overline{u}grad)\overline{u}=\frac{\rho}{(Fr)}\bar{f}-Eu\cdot gradp+$$
$$-\frac{1}{\mathrm{Re}}\left[grad\left(\frac{2}{3}\mu div\overline{u}\right)-div(2\mu[D])\right]$$

In the above equation all parameters are related to their characteristic values, similarly as in the mass conservation equation.

When the N-S equation in the above form is applied to flows in two different scales, we obtain full similarity of the phenomena if all above similarity criteria are fulfilled. This is not always possible. If only some of the criteria are fulfilled, we obtain only partial similarity, and the results of measurements or calculations are subject to so called scale effects (see the example below).

Similarity of flows II

Dimensional analysis of the energy conservation equation

The initial form of the energy conservation equation:

$$\rho'\left[\frac{\partial}{\partial t'}\left(\frac{u'^2}{2}+c'T'\right)+\left(\overline{u}'\bullet grad\right)\left(\frac{u'^2}{2}+c'T'\right)\right]=\rho'\overline{f}'\bullet\overline{u}'-div\left(p'[E]\overline{u}'\right)+$$

$$-div\left(\frac{2}{3}\mu' div\overline{u}'[E]\overline{u}'-2\mu'[D]'\overline{u}'\right)+div\left(\lambda' gradT'\right)$$

It is necessary to introduce the additional scale coefficients:

$$c'=\alpha_c c \qquad T'=\alpha_T T \qquad \lambda'=\alpha_\lambda \lambda$$

The identity of the energy conservation equation in both scales leads to the condition:

$$\frac{\alpha_\rho \alpha_u^2}{\alpha_l}=\frac{\alpha_\rho \alpha_c \alpha_T}{\alpha_t}=\frac{\alpha_\rho \alpha_u^3}{\alpha_l}=\frac{\alpha_\rho \alpha_u \alpha_c \alpha_T}{\alpha_l}=\alpha_\rho \alpha_f \alpha_u=\frac{\alpha_\rho \alpha_u}{\alpha_l}=\frac{\alpha_\mu \alpha_u^2}{\alpha_l^2}=\frac{\alpha_\lambda \alpha_T}{\alpha_l^2}$$

From the above condition we obtain the already known Strouhal, Froude, Eulerand Reynolds numbers, together with two new criteria of similarity:

Eckert number:

$$Ec = \frac{u^2}{cT} = \frac{u'^2}{c'T'}$$

Eckert number defines the ratio of the macroscopic kinetic energy of the fluid motion to the energy of molecular motion (internal energy) of the fluid.

Prandtl number:

$$\mathrm{Pr} = \frac{c\mu}{\lambda} = \frac{c'\mu'}{\lambda'}$$

Prandtl number defines the ratio of the intensity of the fluid momentum transport to the intensity of the fluid energy transport

Prandtl number is the only criterial number composed solely of the material constants.

Using the criterial numbers, the energy conservation equation may be written in the non-dimensional form:

$$Sh\rho\frac{\partial}{\partial t}\left(\frac{u^2}{2}\right)+\frac{Sh}{Ec}\rho\frac{\partial}{\partial t}(cT)+\rho(\bar{u}\bullet grad)\frac{u^2}{2}+\frac{1}{Ec}\rho(\bar{u}\bullet grad)(cT)=$$

$$=\frac{1}{Fr}\rho\bar{f}\bullet\bar{u}-Eu\cdot div(\rho[E]\bar{u})-\frac{1}{\mathrm{Re}}div\left(\frac{2}{3}\mu div\bar{u}[E]\bar{u}-2\mu[D]\bar{u}+\right)$$

$$+\frac{1}{\mathrm{Pr}\cdot\mathrm{Re}\cdot Ec}div(\lambda gradT)$$

All flow parameters in the above equation are related to their characteristic values, what makes them non-dimensional.

Dimensional analysis of the balance of entropy equation

The initial form of the balance of entropy equation:

$$\rho'\left[\frac{\partial e'}{\partial t'} + \overline{u}' \bullet grade'\right] = T'\dot{s}'_m + \frac{p'}{\rho'}\left(\frac{\partial \rho'}{\partial t'} + \overline{u}' \bullet grad\rho'\right) + \lambda'\Delta T$$

The balance of entropy equation does not require introduction of any new scale coefficients. Using the already known scale coefficient we obtain the following identity condition in two different scales:

$$\frac{\alpha_\rho \alpha_c \alpha_T}{\alpha_t} = \frac{\alpha_\rho \alpha_u \alpha_c \alpha_T}{\alpha_l} = \frac{\alpha_\mu \alpha_u^2}{\alpha_l^2} = \frac{\alpha_p}{\alpha_t} = \frac{\alpha_p \alpha_u}{\alpha_l} = \frac{\alpha_\lambda \alpha_T}{\alpha_l^2}$$

This condition does not lead to any new criterial numbers. The balance of entropy equation may be presented in the non-dimensional form using the already known criterial numbers.

The non-dimensional balance of entropy equation:

$$Sh \cdot \rho \frac{\partial}{\partial t}(cT) + \rho(\overline{u} \bullet grad)(cT) = \frac{Ec}{\mathrm{Re}} T\dot{s}_m + Eu \cdot Sh \cdot Ec \frac{p}{\rho}\frac{\partial \rho}{\partial t} +$$

$$+ Eu \cdot Ec \frac{p}{\rho}(\overline{u} \bullet grad)\rho + \frac{1}{\mathrm{Pr} \cdot \mathrm{Re}} \lambda \Delta T$$

Summary

The non-dimensional form of the fluid mechanics equations enables an easy assessment of the relative importance of the respective terms of equations in description of a given flow. The small value of the coefficient composed of the criterial numbers maybe the ground for introducing simplification through removing the respective term from the equation. However, such a simplification should not change the order of the equation. For example, dropping the viscosity terms from the energy conservation equation reduces the order of the equation, which may lead to compromising of the boundary conditions.

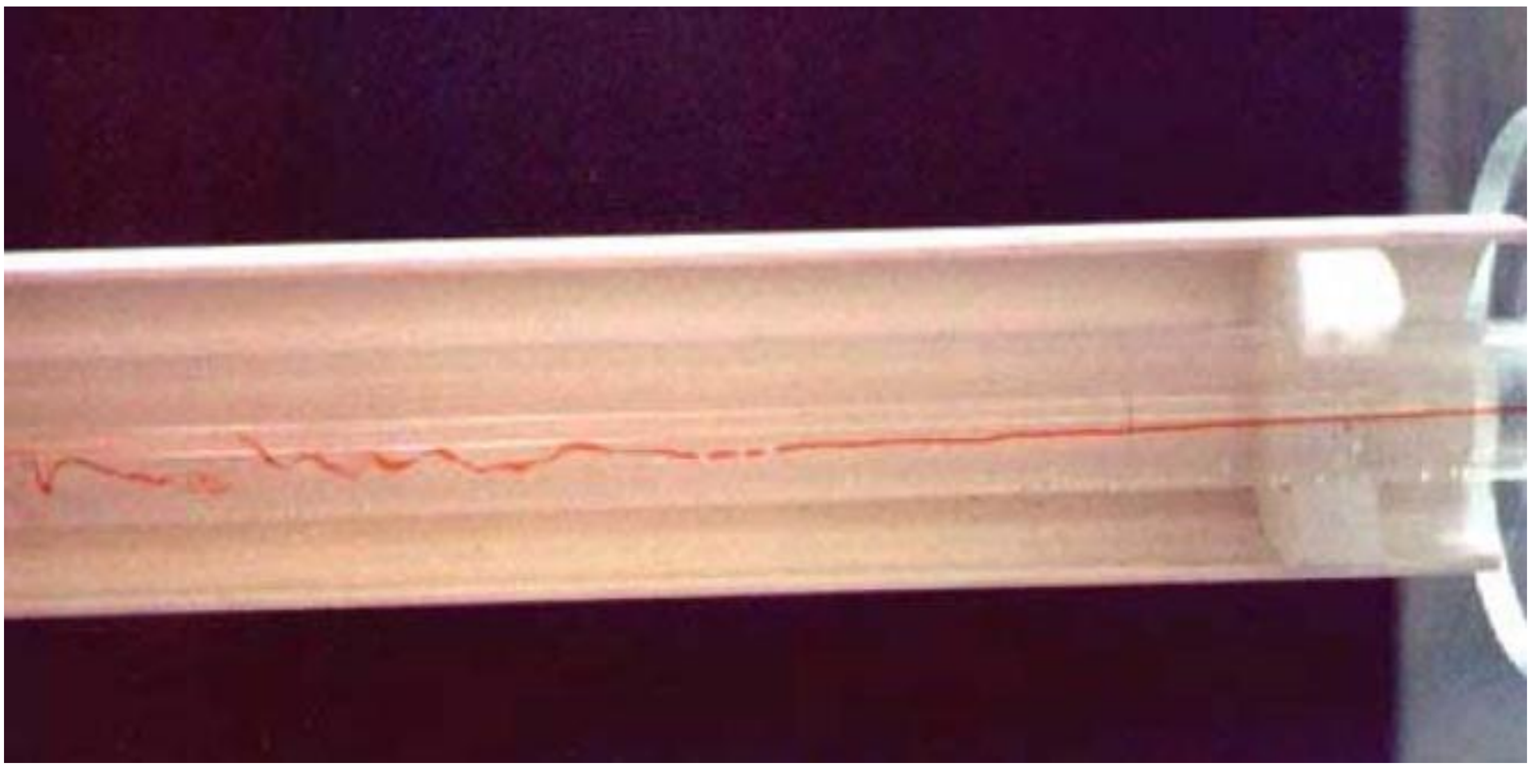
Solution of the non-dimensional fluid mechanics equations has a general form:

$$F\left(Sh, Fr, Eu, \mathrm{Re}, Ec, \mathrm{Pr}\right) = 0$$

If all criterial numbers included in the above formula have identical values for flows in different scales, then these flows are similar to each other.

Chapter 18 – Laminar and turbulent flows

Occurrence of the two distinct types of flow, namely laminar and turbulent, was discovered by Osborne Reynolds in his well-known experiment concerning the flow in a pipe in1883. He came to the conclusion that the laminar flow occurs up to the value of Re=2300. Above that value the fluid motion becomes unstable and intensive mixing of fluid occurs due to vortex structure of the turbulent flow.



$$\mathrm{Re} = \frac{uD}{\upsilon}$$

The ratio of inertia forces and viscosity forces in the fluid flow, expressed by the Reynolds number, influences strongly the character of the flow. At low Reynolds numbers, i.e. with relatively high viscosity forces, the flow has an orderly character – the fluid elements move along parallel tracks and no mutual mixing occurs. Such a flow is called the **laminar flow** or the layered flow. Above a certain value of the Reynolds number (called the **lower critical number**), due to the increasing role of the inertia forces, such a flow loses stability and disturbances exhibiting stochastic fluctuations of the flow velocity appear. With further increase of the Reynolds number (above so called **upper critical number**) the disturbances fill the entire flow, which is then called the **turbulent flow.**

The critical Reynolds numbers are different for different flows, for example they are different for a pipe flow and different for a flow along the plane wall.

The laminar flow–an orderly fluid motion along parallel paths, the fluid elements do not mix with each other, a purely viscous mechanism of transport of momentum and energy dominates the flow.

The turbulent flow – a chaotic fluid motion of a stochastic character, unsteady even with the steady boundary conditions, the fluid elements mix with each other, what leads to an intensive process of transport of mass, momentum and energy.

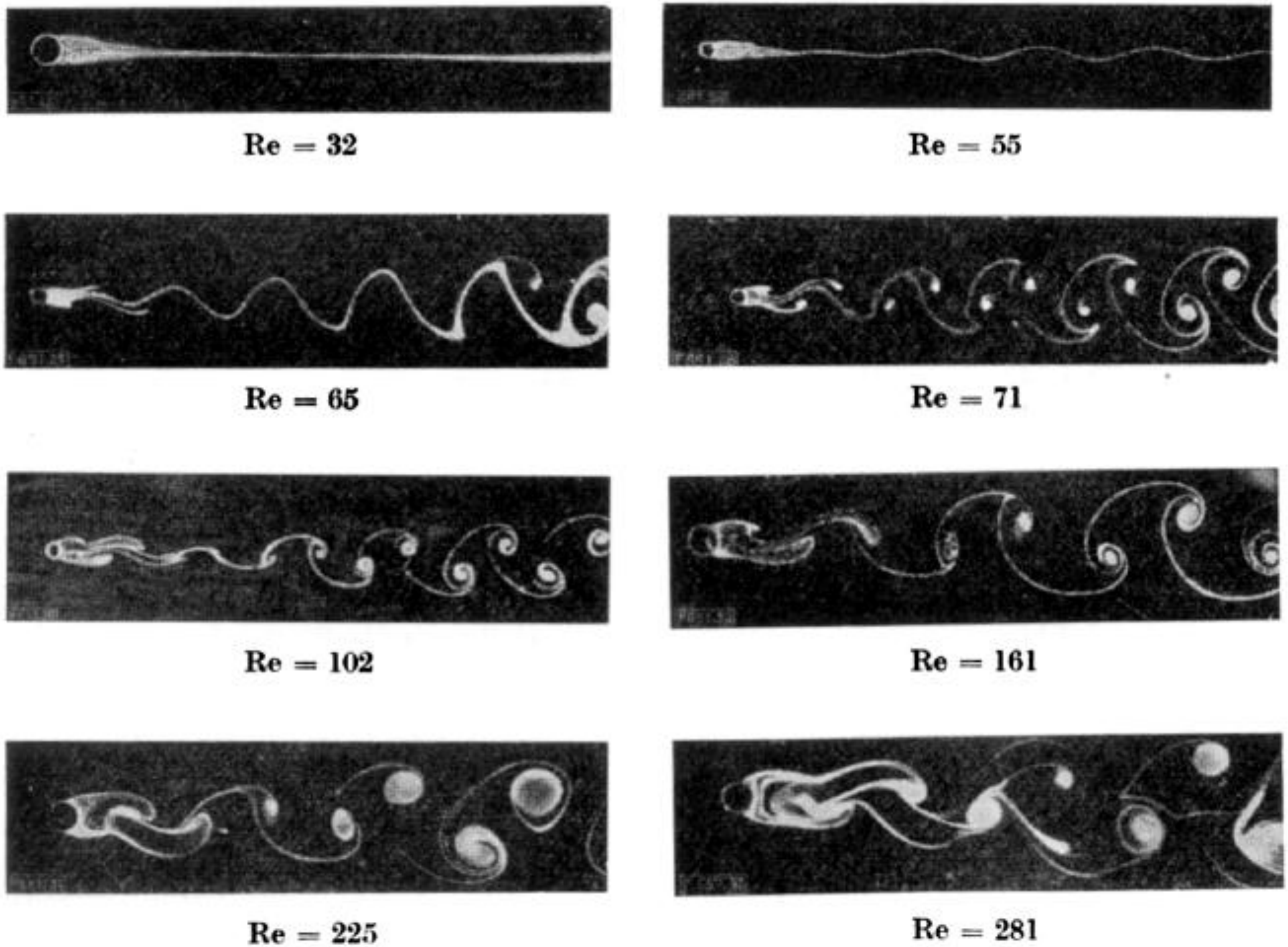
$$\mathrm{Re} = \frac{u \cdot l}{\nu}$$

Re=inertia forces/viscosity forces

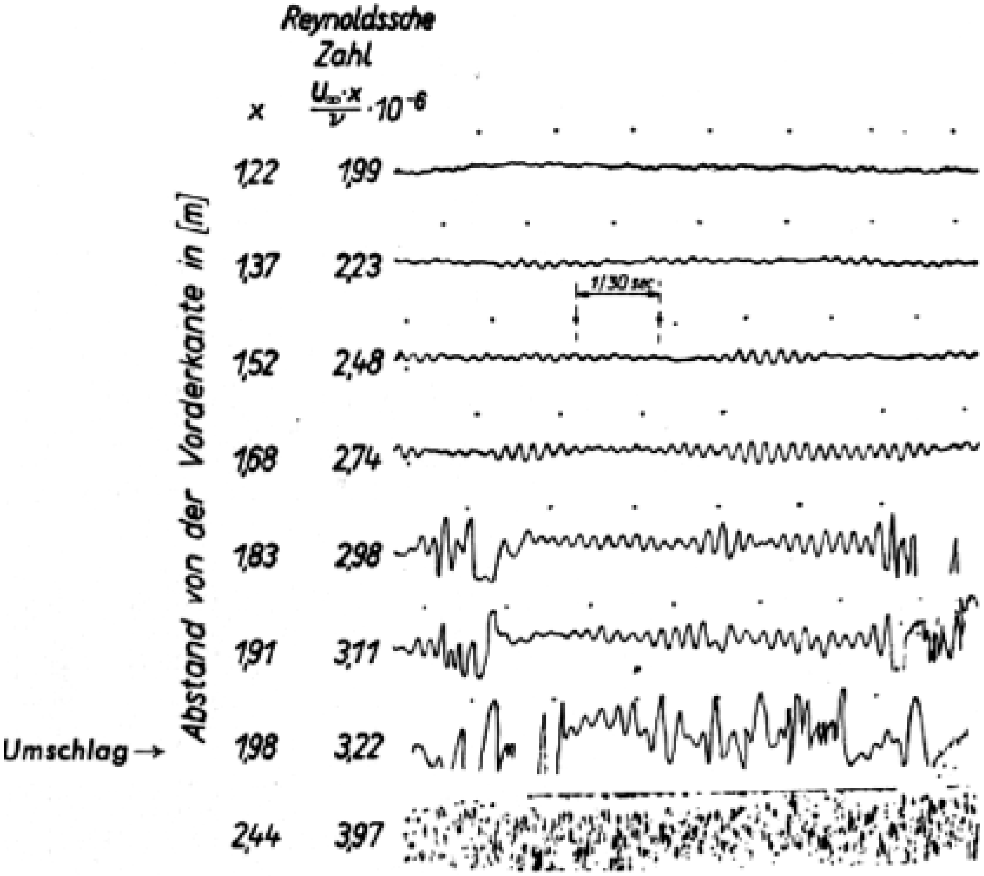
u – the characteristic velocity

L – the characteristic linear dimension

v – the kinematic viscosity coefficient



The above picture shows an experiment concerning the flow around a thin rod, placed perpendicularly to the direction of velocity. The consecutive photographs show the gradual destabilization of the flow as the Reynolds number increases.



The picture shows an increase of the turbulent fluctuations of velocity of the flow along a flat plate, i.e. with the increasing value of the Reynolds number calculated on the basis of the distance from the leading edge of the plate.

For the flow in a pipe of a circular cross-section (D –the diameter) we have:

the lower critical value:

$$\mathrm{Re}_{kr1} = \frac{u \cdot D}{\nu} = 2000$$

the upper critical value:

$$\mathrm{Re}_{kr2} = \frac{u \cdot D}{\nu} = 50000$$

In the flow along a flat plate (x – the distance from the leading edge) we have:

the lower critical value: $$\mathrm{Re}_{kr1} = \frac{u \cdot x_1}{\nu} = 90000$$

the upper critical value: $$\mathrm{Re}_{kr2} = \frac{u \cdot x_2}{\nu} = 1000000$$

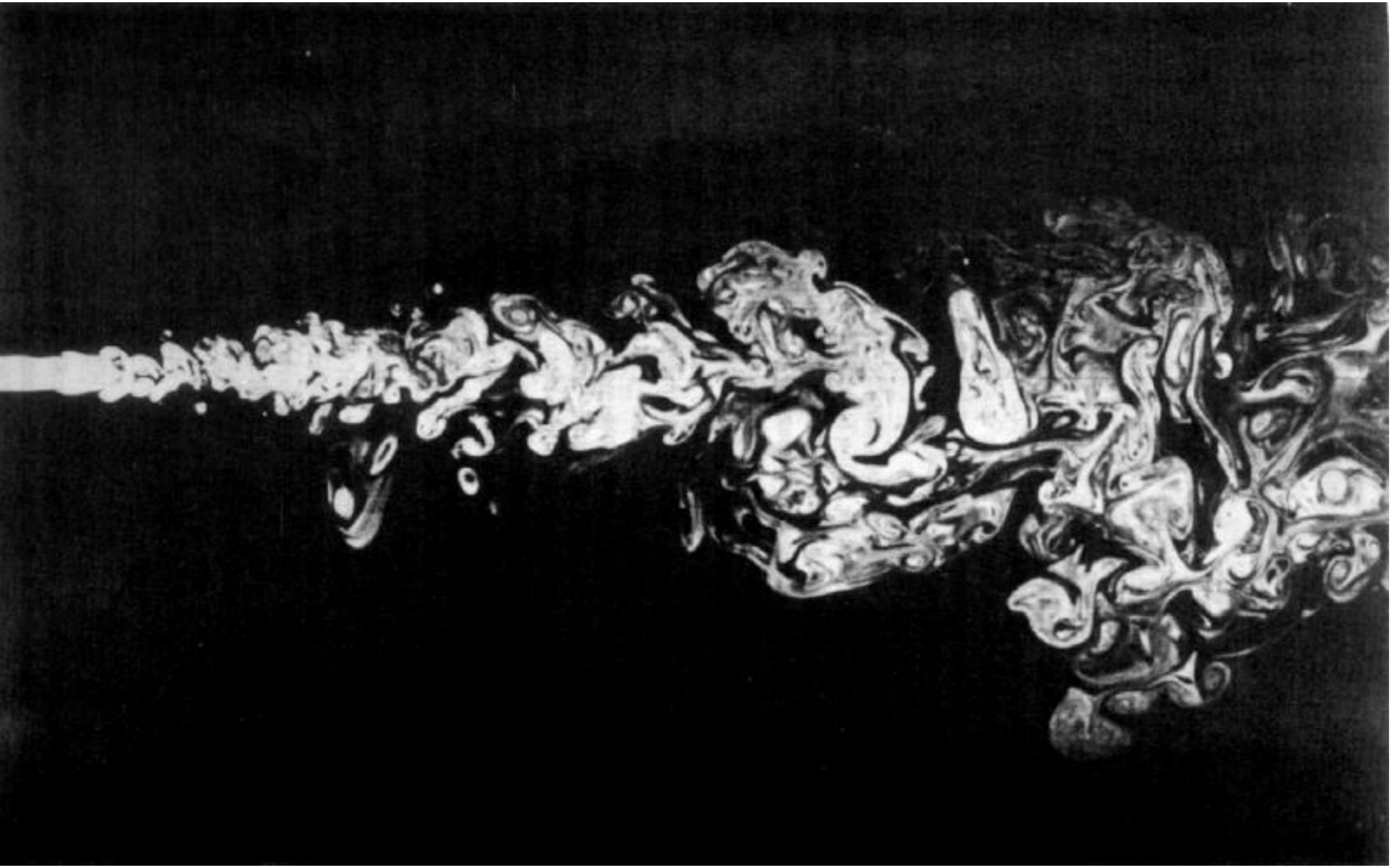
In a turbulent flow we have: $\overline{u} = \overline{U} + \overline{u'}$ or:

The measure of the turbulence intensity is the degree of turbulence ε:
time – dependent velocity = mean velocity + turbulent fluctuation

$$\varepsilon = \frac{\sqrt{\frac{1}{3}\left[(u'_x)^2 + (u'_y)^2 + (u'_z)^2\right]}}{\left|\overline{U}\right|}$$

The kinetic energy of turbulence *k* is given by the expression:

$$k = \frac{1}{2}\left[(u'_x)^2 + (u'_y)^2 + (u'_z)^2\right]$$



Visualization of a turbulent flow shows the specific vortex structures of different scales, known as the turbulent eddies.