Additional Lecture No. 5 - Momentum Analysis of Flow Systems

Introduction

- Fluid flow problems can be analyzed using one of three basic approaches: differential, experimental, and integral (or control volume).
- Control volume forms of the mass and energy equation were developed and used.
- In this chapter, we complete control volume analysis by presenting the integral momentum equation.
  - Review Newton's laws and conservation relations for momentum.
  - Use RTT to develop linear and angular momentum equations for control volumes.
  - Use these equations to determine forces and torques acting on the CV.

Objectives

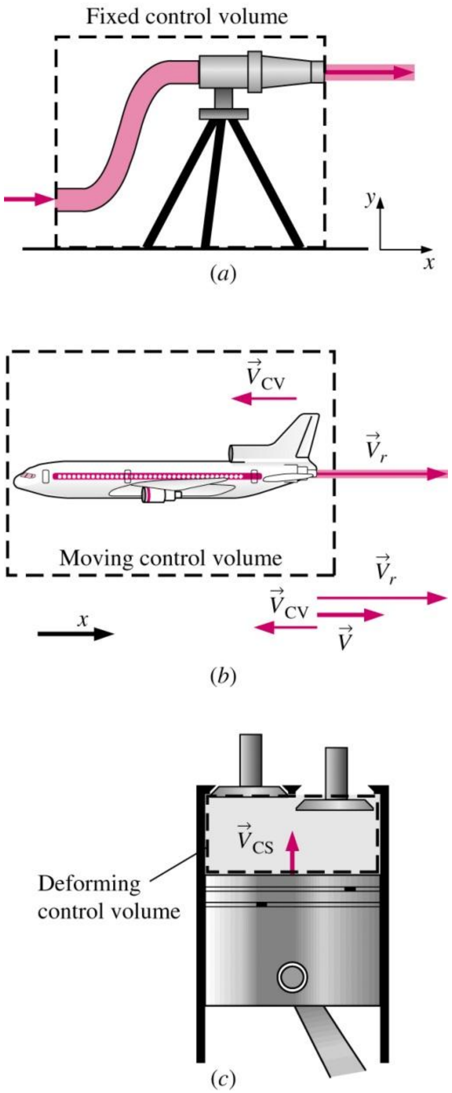
- After completing this chapter, you should be able to
  - Identify the various kinds of forces and moments acting on a control volume.
  - Use control volume analysis to determine the forces associated with fluid flow.
  - Use control volume analysis to determine the moments caused by fluid flow and the torque transmitted.

Newton’s Laws

- Newton’s laws are relations between motions of bodies and the forces acting on them.
  - **First law**: a body at rest remains at rest, and a body in motion remains in motion at the same velocity in a straight path when the net force acting on it is zero.
  - **Second law**: the acceleration of a body is proportional to the net force acting on it and is inversely proportional to its mass.

$$\vec{F} = m\vec{a} = m\frac{d\vec{V}}{dt} = \frac{d\left(m\vec{V}\right)}{dt}$$

  - **Third law**: when a body exerts a force on a second body, the second body exerts an equal and opposite force on the first.



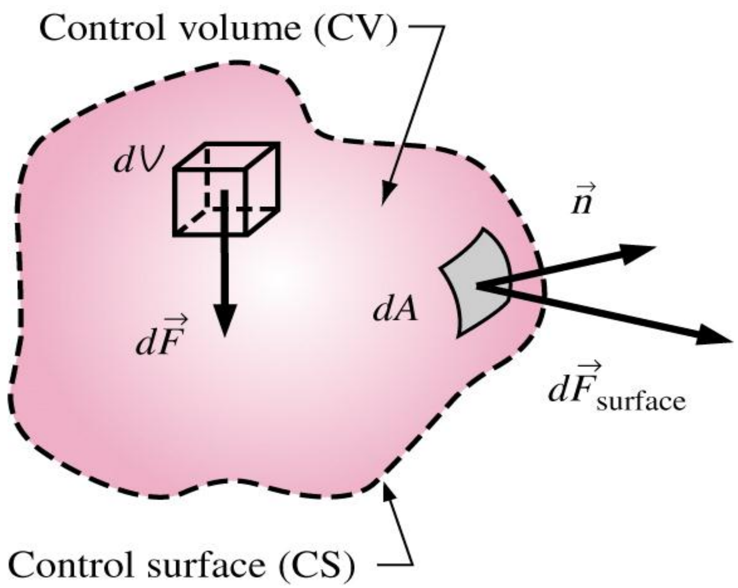
- CV is arbitrarily chosen by fluid dynamicist, however, selection of CV can either simplify or complicate analysis.
  - Clearly define all boundaries. Analysis is often simplified if CS is normal to flow direction.
  - Clearly identify all fluxes crossing the CS.
  - Clearly identify forces and torques *of interest* acting on the CV and CS.
- Fixed, moving, and deforming control volumes.
  - For moving CV, use relative velocity,

$$\vec{V}_r = \vec{V} - \vec{V}_{CV}$$

  - For deforming CV, use relative velocity all deforming control surfaces,

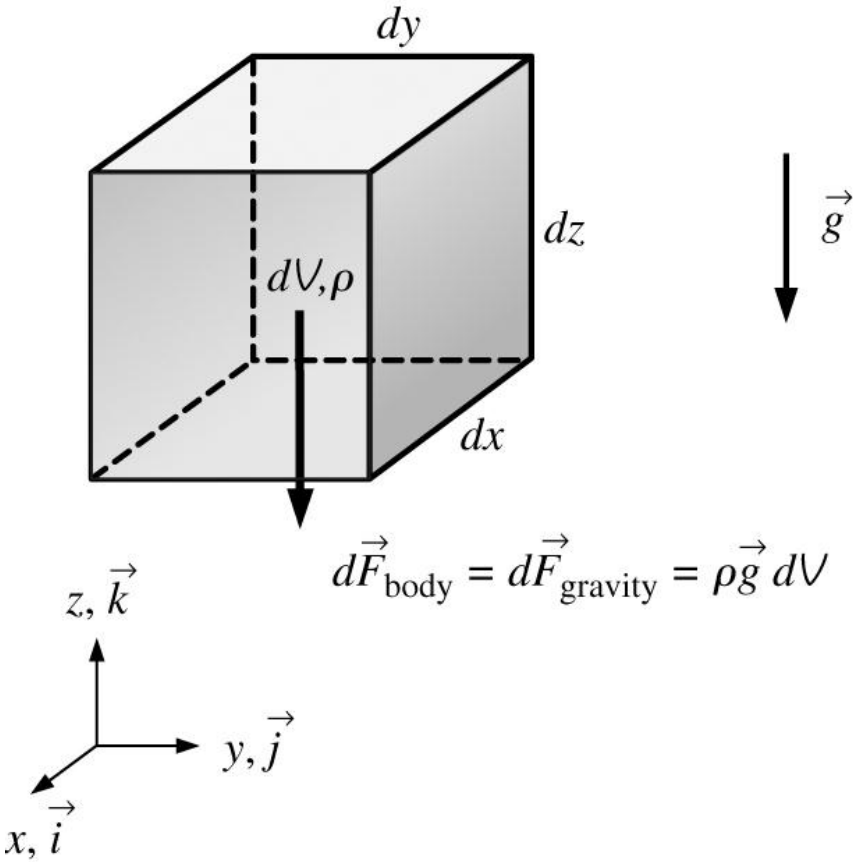
$$\vec{V}_r = \vec{V} - \vec{V}_{CS}$$

- Forces acting on CV consist of **body forces** that act throughout the entire body of the CV (such as gravity, electric, and magnetic forces) and **surface forces** that act on the control surface (such as pressure and viscous forces, and reaction forces at points of contact).



- Body forces act on each volumetric portion *dV* of the CV.
- Surface forces act on each portion *dA* of the CS.

Body Forces



- The most common body force is gravity, which exerts a downward force on every differential element of the CV
- The different body force

$$d\vec{F}_{body} = d\vec{F}_{gravity} = \rho\vec{g}d\mathcal{V}$$

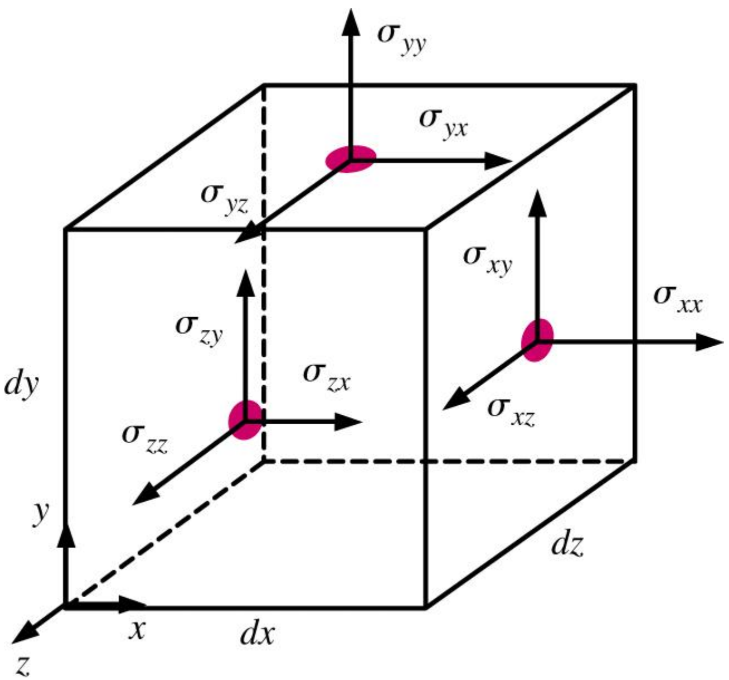
- Typical convention is that $\vec{g}$ acts in the negative z-direction,

$$\vec{g} = -g\vec{k}$$

- Total body force acting on CV

$$\sum \vec{F}_{body} = \int_{CV} \rho\vec{g}d\mathcal{V} = m_{CV}\vec{g}$$

Surface Forces

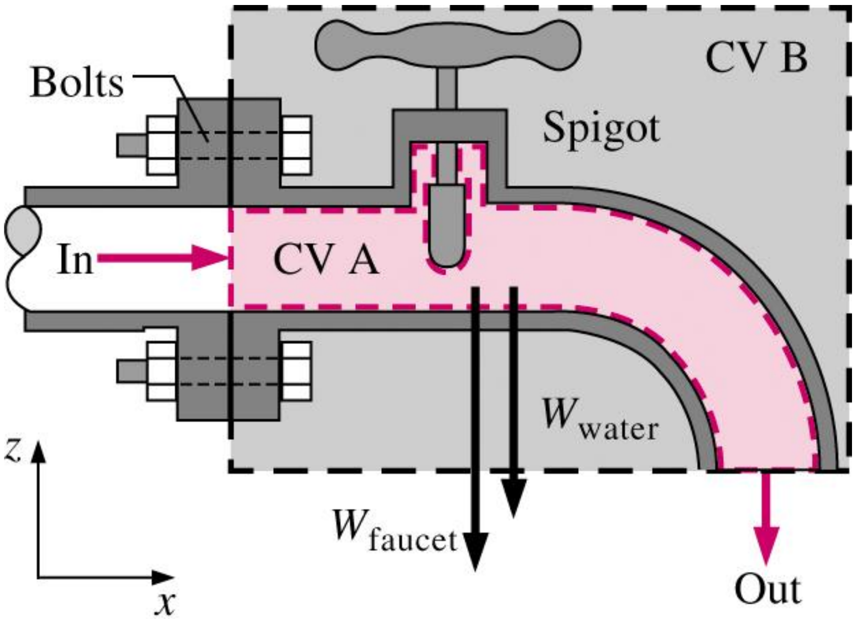


- Surface forces are not as simple to analyze since they include both normal and tangential components
- Diagonal components σ_{xx}, σ_{yy}, σ_{zz} are called **normal stresses** and are due to pressure and viscous stresses
- Off-diagonal components σ_{xy}, σ_{xz}, etc., are called **shear stresses** and are due solely to viscous stresses
- Total surface force acting on CS

$$\sigma_{ij} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix}$$

$$\sum \vec{F}_{surface} = \int_{CS} \sigma_{ij} \cdot \vec{n} dA$$

Body and Surface Forces



- Surface integrals are cumbersome.
- Careful selection of CV allows expression of total force in terms of more readily available quantities like weight, pressure, and reaction forces.
- Goal is to choose CV to expose only the forces to be determined and a minimum number of other forces.

$$\sum \vec{F} = \underbrace{\sum \vec{F}_{gravity}}_{\text{body force}} + \underbrace{\sum \vec{F}_{pressure} + \sum \vec{F}_{viscous} + \sum \vec{F}_{other}}_{\text{surface forces}}$$

Linear Momentum Equation

- Newton's second law for a system of mass *m* subjected to a force $\vec{F}$ is expressed as

$$\vec{F} = m\vec{a} = m\frac{d\vec{V}}{dt} = \frac{d}{dt}\left(m\vec{V}\right)$$

- Use RTT with $b = \vec{V}$ and $B = m\vec{V}$ to shift from system formulation of the control volume formulation

$$\frac{d\left(m\vec{V}\right)_{sys}}{dt} = \frac{d}{dt}\int_{CV} \rho\vec{V}\, d\mathcal{V} + \int_{CS} \rho\vec{V}\left(\vec{V}_r \cdot \vec{n}\right)\, dA$$

$$\sum \vec{F} = \frac{d}{dt}\int_{CV} \rho\vec{V}\, d\mathcal{V} + \int_{CS} \rho\vec{V}\left(\vec{V}_r \cdot \vec{n}\right)\, dA$$

Special Cases

- Steady Flow $$\sum \vec{F} = \int_{CS} \rho \vec{V} \left(\vec{V}_r \cdot \vec{n} \right) \, dA$$
- Average velocities $$\dot{m} = \int_{A_C} \rho \left(\vec{V} \cdot \vec{n} \right) \, dA_c = \rho V_{avg} A_c$$
- Approximate momentum flow rate

$$\int_{CS} \rho \vec{V} \left(\vec{V} \cdot \vec{n} \right) \, dA \approx \rho V_{avg} A_c \vec{V}_{avg} = \dot{m} \vec{V}_{avg}$$

- To account for error, use momentum-flux correction factor β

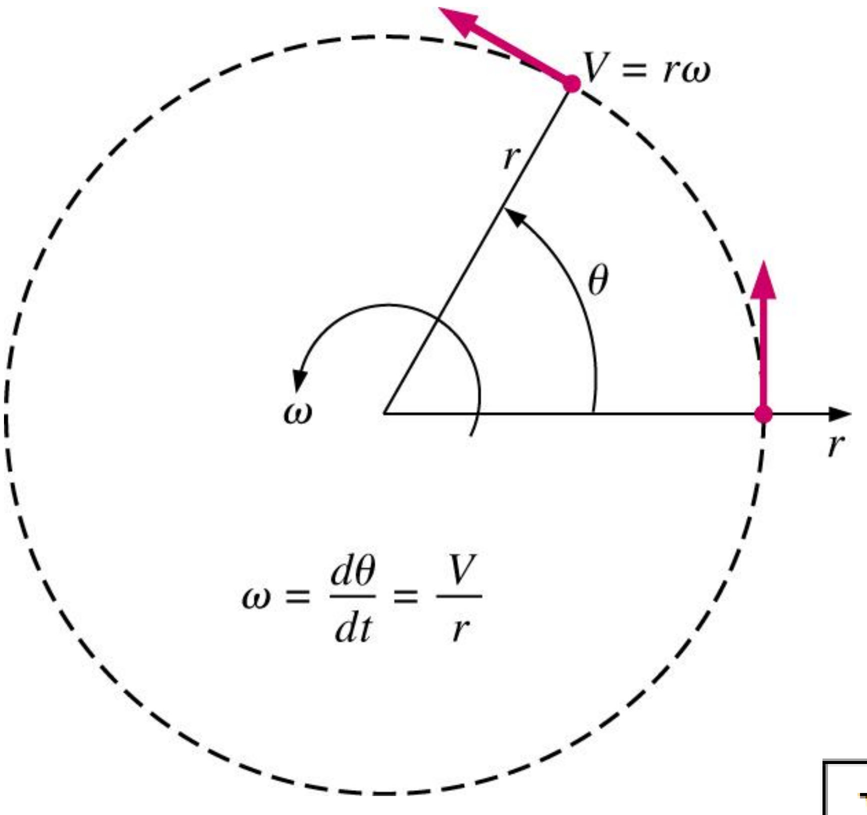
$$\sum \vec{F} = \frac{d}{dt} \int \rho \vec{V} \, d\mathcal{V} + \sum_{out} \beta \dot{m} \vec{V}_{avg} - \sum_{in} \beta \dot{m} \vec{V}_{avg}$$

$$\beta = \frac{1}{A_c} \int_{A_C} \left(\frac{V}{V_{avg}} \right)^2 dA_c$$

Angular Momentum

- Motion of a rigid body can be considered to be the combination of
  - the translational motion of its center of mass (U_x, U_y, U_z)
  - the rotational motion about its center of mass (ω_x, ω_y, ω_z)
- Translational motion can be analyzed with linear momentum equation.
- Rotational motion is analyzed with angular momentum equation.
- Together, the body motion can be described as a 6–degree–of–freedom (6DOF) system.

Review of Rotational Motion



Angular velocity ω is the angular distance θ traveled per unit time, and angular acceleration α is the rate of change of angular velocity.

$$\omega = \frac{d\theta}{dt} = \frac{d(l/r)}{dt} = \frac{1}{r}\frac{dl}{dt} = \frac{V}{r}$$

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} = \frac{1}{r}\frac{dV}{dt} = \frac{a_t}{r}$$

$$\boxed{V = r\omega \text{ and } a_t = r\alpha}$$

Review of Angular Momentum

- Moment of a force: $\vec{M} = \vec{r} \times \vec{F}$
- Moment of momentum: $\vec{H} = \vec{r} \times m\vec{V}$
- For a system: $\vec{H}_{sys} = \int_{sys} (\vec{r} \times \vec{V})\rho \, d\mathcal{V}$

$$\frac{d\vec{H}_{sys}}{dt} = \frac{d}{dt}\int_{sys} (\vec{r} \times \vec{V})\rho \, d\mathcal{V}$$

- Therefore, the angular momentum equation can be written as: $\sum \vec{M} = \frac{d\vec{H}_{sys}}{dt}$
- To derive angular momentum for a CV, use RTT with $B = \vec{H}$ and $\beta = \vec{r} \times \vec{V}$

Angular Momentum Equation for a CV

- General form

$$\sum \vec{M} = \frac{d}{dt}\int_{CV} (\vec{r} \times \vec{V})\rho \, d\mathcal{V} + \int_{CS} (\vec{r} \times \vec{V})\rho \left(\vec{V}_r \cdot \vec{n}\right) \, dA$$

- Approximate form using average properties at inlets and outlets

$$\sum \vec{M} = \frac{d}{dt}\int_{CV} (\vec{r} \times \vec{V})\rho \, d\mathcal{V} + \sum_{out} \vec{r} \times \dot{m}\vec{V} - \sum_{in} \vec{r} \times \dot{m}\vec{V}$$

- Steady flow

$$\sum \vec{M} = +\sum_{out} \vec{r} \times \dot{m}\vec{V} - \sum_{in} \vec{r} \times \dot{m}\vec{V}$$